# MATHEMATICS

## STUDENT'S TEXTBOOK GRADE 10

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**FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA**
**MINISTRY OF EDUCATION**

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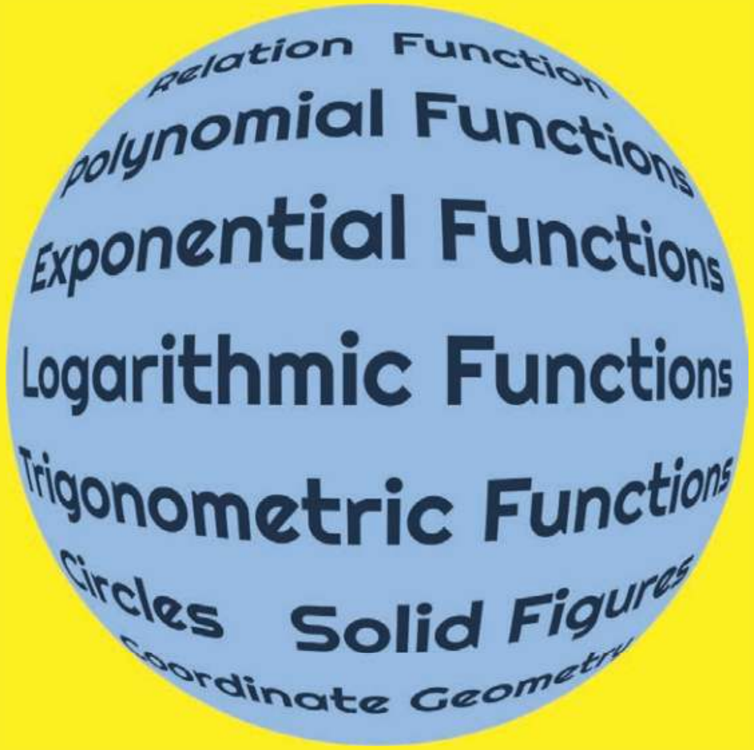


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# MATHEMATICS

## STUDENT TEXTBOOK GRADE 10

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## Welcoming Message to Students.

Dear grade 10 students, you are welcome to the second grade of secondary level education, which is a golden opportunity in your academic career. This is a continuation and advancement of grade 9 Mathematics education. In this stage, you are expected to get more advanced knowledge and experiences which can help you enhance your academic, social, and personal growth in the field of Mathematics. You, therefore, need to bring your textbook to class and practice exercises regularly.

Enjoy it!

## Introduction on Students' Textbook.

Dear students, this textbook has 7 units namely: Relations and Functions, Polynomial functions, Exponential and Logarithmic Functions, Trigonometric functions, Circles, Solid Figures and Coordinate Geometry respectively. Each of the units is composed of introduction, objectives, lessons, key terms, summary, and review exercises. Each unit is basically unitized, or lesson based, and each lesson has four components: Activity, Definition, Examples, and exercises (ADEE).

The most important part in this process is to practice problems by yourself based on what your teacher shows and explains. Your teacher will also give you feedback, assistance, and facilitate further learning. In such a way, you will be able to not only acquire new knowledge and skills but also develop them further.

## Activity

This part of the lesson demands you to revise what you have learnt or activate your background knowledge on the topic. The activity also introduces you to what you are going to learn in a new lesson topic.

## Definition/Theorem/Note

This part presents and explains to you new concepts. However, every lesson may not begin with definition, especially when the lesson is a continuation of the previous one.

## Example and Solution

Here, your teacher will give you specific examples to improve your understanding of the new content. In this part, you need to listen to your teacher's explanation carefully and participate actively. Note that your teacher may not discuss all of the examples in the class. In this case, you need to attempt and internalize the examples by yourself.

## Exercise

Under this part of the material, you will solve the exercises and questions individually, in pairs or groups to practice what you learnt in the examples. When you are doing the exercise in the classroom either in pairs or groups, you are expected to share your opinions with your friends, listen to others' ideas carefully and compare yours with others. Note that you will have the opportunity of cross checking your answers to the questions given in the class with the answers of your teacher. However, for the exercises not covered in the class, you will be given as a homework, assignment, or project. In this case, you are expected to communicate your teacher for the solutions.

# Contents

# UNIT 1

# RELATIONS AND FUNCTIONS

## Unit Outcomes

By the end of this unit, you will be able to:

- **Define relation.**
- **Define function.**
- **Identify types of functions.**
- **Sketch graphs of various types of relations and functions.**

## Unit Contents


- ✓ leading coefficient
- ✓ vertex
- ✓ linear function
- ✓ x-intercept
- ✓ quadratic function
- ✓ y-intercept
- ✓ constant function
- ✓ relation
- ✓ parabola
- ✓ coordinate system
- ✓ range
- ✓ turning point
- ✓ combination of functions
- ✓ slope
- ✓ Function
- ✓ axis(orthogonal-axis)
- ✓ axis of symmetry
- ✓ domain

# 1.1 Relations

## Introduction:

In order to continue our study of functions, we introduce the more general idea of a relation. As its name suggests, the concept of a relation is a familiar one. In our daily life, we come across many patterns that characterize relations with brothers and sisters, mother and daughters, father and sons, teachers and students etc. In mathematics also, we come across many relations such as number $m$ is greater than number $n$; line $n$ is perpendicular to line $m$ etc... The concept of relation is established in mathematical form. The word "function" is introduced by Leibnitz in $17^{th}$ century. ***Function is defined as a special type of relation.*** In the present unit, we shall discuss Cartesian coordinates, conditions for a relation to be a function, different types of functions and their properties.

## 1.1.1 Revision of patterns

### Activity 1.1

1. Write the numbers which come next in $1, 3, 5, 7, 9,$ ______
2. Which of the following arrows in figure1.1 under column four can fill in the blank space under the column three to continue the service below?

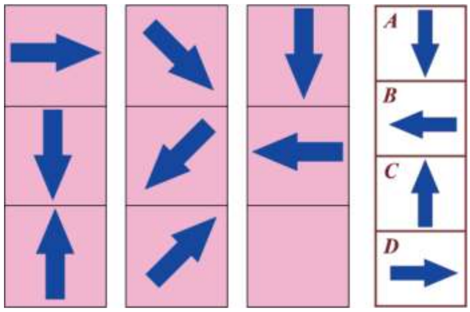



Figure 1.1

3. Create a table of values for the set of values given in the first row of Table1.1 by evaluating the algebraic expression $2n + 3$.

Table 1.1

| Input $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Output $= 2n + 3$ | | | | | | |

4. What relationship can be represented by the table given below?

Table 1.2

| Input | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Output | 2 | 5 | 8 | 11 | 14 |

A **pattern** is a regularity in the world in human-made design or in abstract ideas. As such, the elements of a **pattern** repeat in a predictable manner.

Patterns are defined as regular, repeated, recurring forms or designs identifying relationships, finding logic to form generalizations and make predictions.

## Example 1

Even numbers pattern: 2, 4, 6, 8 …

Odd numbers pattern: 1, 3, 5, 7, 9 …

Arithmetic pattern

The arithmetic pattern is also known as the algebraic pattern. In an arithmetic pattern, the sequences are based on the addition or subtraction of the terms. If two or more terms in the sequence are given, we can use addition or subtraction to find the arithmetic pattern.

For example, consider the pattern 2, 4, 6, 8, 10, __, 14, __. Now, we need to find the missing term in the pattern.

Here, we can use the addition process to figure out the missing terms in the patterns.

In the pattern, the rule used is “Add 2 to the previous term to get the next terms”.

First missing term: The previous term is 10. Therefore, 10+2 = 12.

Second missing term: The previous term is 14. So, 14+2 = 16

Hence, the complete arithmetic pattern is 2, 4, 6, 8, 10, **12**, 14, **and 16**.

Geometric pattern

The geometric pattern is defined as the sequence of numbers that are based on the multiplication and division operation. Similar to the arithmetic pattern, if two or more numbers in the sequence are provided, we can easily find the unknown terms in the pattern using multiplication and division operation.

For example, consider the pattern 2, 4, 8, __, 32, __.

It is a geometric pattern, as each term in the sequence can be obtained by multiplying 2 with the previous term. 8 is the third term in the sequence, which is obtained by multiplying 2 with the previous term 4.

First missing term: The previous term is 8. Multiply 8 by 2, we get 16.

Second missing term: The previous term is 32. Multiply 32 by 2, we get 64.

Hence, the complete geometric pattern is 2, 8, **16**, 32, **and 64**.

## Exercise 1.1

1. Fill the blank spaces using the following pattern.

   65, 60, 55, 50, 45, ___, 35, ___.
2. Identify the type of pattern for the sequence 4, 8, 12, 16, 20, …
3. Fill the bank spaces using the following pattern.

   15, 22, 29, 36, 43, ___, 57, 64, 71, 78, 85, ___.

4. Find the missing value for the geometric pattern: $96, 48, 24, \_\_, 6, \_\_$.

5. In figure1.2, how many sticks are in the next diagram? How many sticks are in the tenth diagram? Write an algebraic expression to describe this system?

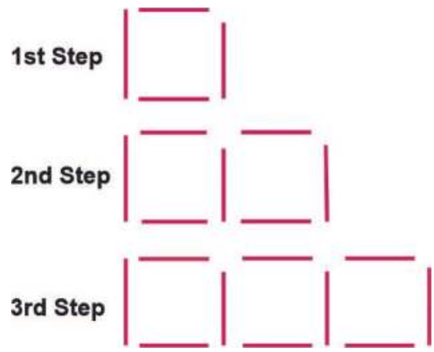
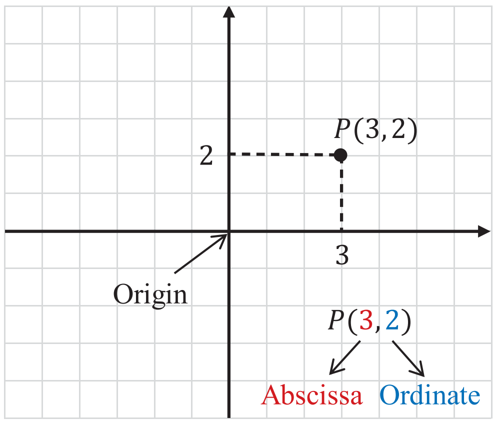



Figure 1.2 Linear patterns

## 1.1.2 Cartesian coordinate system in two dimensions

The Cartesian coordinate system in two dimensions (also called a rectangular coordinate system) is defined by an ordered pair of perpendicular lines (axes). It has a single unit of length for both axes and an orientation for each axis. The point where the axes meet is taken as the origin for both and used as a turning point for each axis into a number line. For any point (P), a line drawn through $P$ perpendicular to each axis and the position where it meets the axis is interpreted as a number. The two numbers in that chosen order are the Cartesian coordinate of P. The first and second coordinates are called the abscissa and the ordinate of P, respectively and the point where the axes meet is called the origin of the coordinate system. The coordinates are usually written as two numbers in parentheses in that order separated by a comma, as in $(3, 2)$. Thus the origin has coordinates $(0, 0)$, and the points on the positive half-axes which is one unit away from the origin and have coordinates $(1, 0)$ and $(0, 1)$.




In mathematics, physics, and engineering, the first axis is usually defined or depicted as horizontal and oriented to the right. The second axis is vertical and oriented upwards. The origin is often labeled O, and the two coordinates are often denoted by the letters ***X*** and ***Y***, or ***x*** and ***y***. The axes may then be referred to as the  and  -axis. The choices of letters come from the original convention which is to use the latter part of the alphabet to indicate unknown values. Therefore, this coordinate system is also called  -coordinate. A plane with coordinate axes is called  -plane. The coordinate plane is divided into 4 parts, namely Quadrant I, II, III, and IV as shown in the figure1.3.

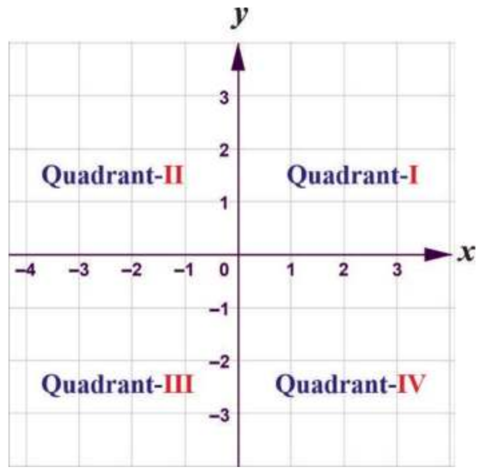



Figure 1.3

## Activity 1.2

1. Complete the following table with +, − or 0 that applies to each coordinate of a point *P* ( , ).

Table 1.3

| **Quadrants** | axes | |
|---|---|---|
| | | |
| Quadrant-I | | |
| Quadrant-II | | |
| Ouadrant-III | | |
| Ouadrant-IV | | |
| Positive - *axis* | | |
| Negative -axis | | |
| Positive -axis | | |
| Negative -axis | | |

2. Plot the points $P(-2,\ 2)$, $\mathrm{Q}(2,4)$, $\mathrm{R}(0,-3)$, $\mathrm{S}(-2,1)$ and $\mathrm{T}(-5,-3)$ on the coordinate. system.

A system in which the location of a point is given by coordinates that represent its distances from perpendicular lines that intersect at a point called the origin. A Cartesian coordinate system in a plane has two perpendicular lines (the $x$-axis and $y$-axis).

In mathematics, the **Cartesian coordinate system** (or **rectangular coordinate system**) is used to determine each point uniquely in a plane through two numbers, usually called the -coordinate and the -coordinate of the point. To define the coordinates, two perpendicular directed lines (the -axis or abscissa, and the -axis or ordinate), are specified, as well as the unit length, which is marked off on the two axes (see Figure1.4).

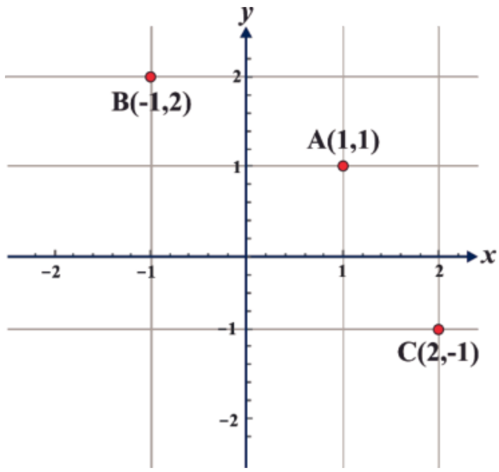



Figure 1.4 the -plane

Figure 1.4 shows the location of the point $A\ (1, 1)$ in the -plane. You may note that the position of the ordered

pair $B\ (-1,\ 2)$ is different from that of $C\ (2, -1)$ Thus, we can say that ( , ) and ( , ) are two different ordered pairs representing two different points in a plane.

## Exercise 1.2

Plot the points whose coordinates are given on a Cartesian coordinate system. $P\ (-3, -5),\ Q\ (-4, 3),\ R\ (0, 2),\ S\ (-2, 0)$.

## 1.1.3 Basic concepts of relations

In order to continue our study of functions, we introduce the more general idea of a relation, as its name suggests, the concept of relation is a familiar one. Everyone has relatives or relations -father, mother, brothers, and sisters etc. What may come as a surprise, however, is that this concept has an important place in mathematics. Let us start with an example taken from everyday life in the relationship of fatherhood. The problem we set for ourselves is to describe this relation in mathematical terms.

If you think about this problem for a few minutes, you will probably find it somewhat more difficult than you expected. Although everyone knows what it means to say that  is a father of  , it is not quite so clear how to put this in the language we use in mathematics.

In our daily life, we usually talk about relations between various things. For example, 7 is less than 9; Addis Ababa is the capital city of Ethiopia; Walia Ibex is endemic to Ethiopia, and so on. In all these cases, we find that a relation involves pairs of objects in some specific order. In this unit, you will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. You also learn here about special relations which will qualify to be functions.

## Activity 1.3

Let set $A$ contains the elements 1, 2, 4, 6, 7 and set $B$ contains the elements 3, 5, 7, 8, 9, 12. List all the ordered pairs ( , ) which satisfy each of the following sentences where  is an element of $A$ and  is an element of $B$.

a.  is greater than  .

b. The sum of  and  is odd.

c.  is a multiple of  .

d.  is half of  .

**Note**

In activity 1.1 we have observed the following:

1. In the case of relations between objects and patterns, order is important.
2. A relation establishes pairing between objects.

Therefore, from a mathematical standpoint, the meaning of a relation is given below.

## Definition 1.1

A relation is a set of ordered pairs. It is denoted by $R$.

## Example 1

Given a relation $R$: The set of all ordered pairs ( , ) of real numbers where  is greater than ,

1. Which of the following ordered pairs belong to this relation?

   $(2,4), (4,3), (1.1, 1.11), (1.1, -3), (-5,-3),\ (7,7), (\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{3}{2}), (0.45, 0.46)$

2. Find a number $c$ such that

   **a.** $(4, c)$ is in the relation;

   **b.** $(-4, c)$ is in the relation;

   **c.** $(4, c)$ and $(c, 4)$ are both in the relation.

**Solution:**

1. $(2,4), (1.1, 1.11), (-5,-3), (\frac{2}{3}, \frac{3}{2})$ and $(0.45, 0.46)$ belongs to the relation $R$.

2. **a.** the set of real numbers $c > 4$

   **b.** the set of real numbers $c > -4$

   **c.** there is no real number $c$ that satisfies the relation $R$.

Is there more than one answer to each of these questions? Why? As seen from the above example 1, question number 1 has only one solution, question number 2, a and b have many solutions and question number 2c has no solution. Since an ordered pair of real numbers can be pictured as a point on the graph, we can use graphs to represent relations $R$ as:

$R$ is the set containing ordered pairs such that $(-3,-3), (-3,3), (-2,2), (-1,2),$ $(1,2), (3,2), (-2,-2), (2.6, 0.6), (1, 0.6)$

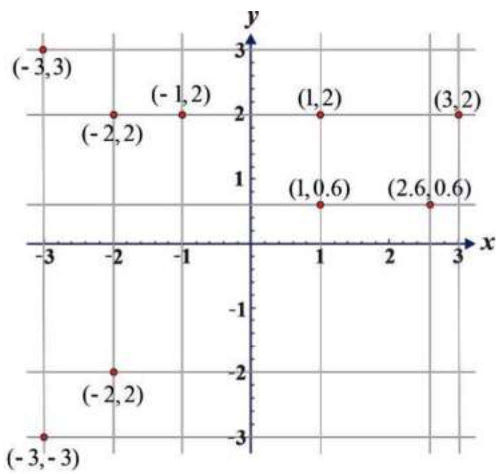



Figure 1.5 Graph of relation $R$

## Exercise 1.3

Given a relation $R$ of set of all ordered pairs ( , ) of real numbers where   is less than  .

a. Which of the following ordered pairs belong to the relation?

$(2, 1), (-4, 3), (-2, 0), (0.2, 0.21), (-0.2, -0.21), (7, 7), (-2, -3), (0, -5)$

b. Find a number $n$ such that  (i) $(n, 0)$  (ii) $(0, n)$

### Example 2

Let $R$ be a relation of the set of all ordered pairs ( , ) of natural numbers where   is a multiple of  , then which of the following ordered pairs belong to $R$ ?

$(2, 4), (4, 3), (3, 9), (18, -3), (9, 3), (7, 7), (3, 12), (6, 18), (30, 5)$

**Solution:**

$(2, 4), (3, 9), (7, 7), (3, 12)$ and $(6, 18)$ belong to the relation $R$.

### Example 3

1. Let $R$ denote the set of ordered pairs ( , ) of real numbers, where $= \ ^2$.
   a. Find the ordered pairs belong to $R$ which have the following first entries: $0, 1, -1, -2, \frac{1}{5}, 3, -3$.
   b. Find the ordered pairs belong to $R$ which have the following second entries: $4, 1, 0, \frac{1}{4}, \frac{1}{25}$.

**Solution:**

a. $(0,0), (1,1), (-1,1), (-2,4), (\frac{1}{5}, \frac{1}{25}), (3,9), (-3,9)$.

b. $(2,4), (1,1), (0,0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{5}, \frac{1}{25})$.

## Exercise 1.4

1. If $R$ is a relation of a set of ordered pairs ( , ) of real numbers such that $= 3 \ - 2$, then list some of the ordered pairs belong to $R$.
2. Let $R$ denote the set of ordered pairs ( , ) of real numbers, where $= \ ^3$.

**a.** Find the ordered pairs belong to $R$ which have the following first entries: $0, 1, 2, -2, 8, \frac{1}{5}, 3, -3$.

**b.** Find the ordered pairs belong to $R$ which have the following second entries: $8, -1, -8, -27, \frac{1}{27}$.

## Domain and range of a relation

Any set of ordered pairs ( , ) is called a relation in and y.

The set of first components in the ordered pairs is called the **domain** of the relation.

The set of second components in the ordered pairs is called the **range** of the relation.

### Activity 1.4

For the relation $R$ of the set of ordered pairs $(5, 3), (-2, 4), (5, 2), (-2, 3)$ determine the domain and the range.

In Activity 1.4, the first ordered pairs of $R$ are 5, -2 and the second ordered pairs of $R$ are 2, 3, 4

### Definition 1.2

Let $R$ be a relation from a set A to a set B. Then

i. Domain of $R$ = { : ( , ) belongs to R for some }

ii. Domain of $R$ = { : ( , ) belongs to R for some }

### Example 1

Determine the domain and range of the relation with ordered pairs $(-2, 1), (-1, 0), (0, 0), (4, 2), (3, 5)$.

**Solution:**

The domain of the relation is -2, -1, 0, 4 and 3. The range of the relation are 1, 0, 2 and 5.

### Example 2

Write some ordered pairs that belong to the relation $R$ which contains the set of ordered pair ( , ) such that = 2 , and are members of integers. Find also the domain and range of $R$.

**Solution:**

$R$ contains some set of ordered pairs $(-2,-4),(-1,\ -2),(0,0),(1,2),(2,4)$.

The domain of the relation R is the set of all integers and the range of the relation R is the set of all even integers.

## Example 3

Find the domain and the range of each of the following relations:

**a.** $R$ is the set of ordered pairs ( , ) such that is the square root of .

**b.** $R$ is the set of ordered pairs ( , ) such that is the square of .

**Solution:**

**a.** Domain is the set of real numbers : is greater than or equal to zero and Range is the set of real numbers : is greater than or equal to zero.

**b.** Domain is the set of all real numbers and Range is the set of all real numbers : is greater than or equal to zero.

## Exercise 1.5

1. Determine the domain and range of the relation with ordered pairs $(-1,\ 4)$, $(0,\ 7),(2,\ 3),(3,\ 3),(4,-2)$.
2. Write some ordered pairs that belongs to the following relation $R$; and find also the domain and range of this relation.
   a. The set of ordered pair ( , ) such that = 3 : and are members of integers.
   b. The set of ordered pair ( , ) such that = −2 : and are members of integers.
3. Let $R$ be a relation of the set of ordered pairs ( , ) of real numbers such that the sum of whose squares is one.
   a. Identify the ordered pairs which belong to R:

$(1,1), (0,1), (0,-1), (2,-1), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), (\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (0,0), (-\frac{\sqrt{3}}{2}, -\frac{1}{2}),$

$(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

b. Find the domain and range of R.

## 1.1.4 Graphs of Relations

By now, you have understood what a relation is and how it can be described using Cartesian coordinates. You will now see how relations can be represented through graphs. You may graphically represent a relation *R* from domain to range by locating the ordered pairs in a coordinate system.

Discuss the following:

i. A Cartesian coordinate system (or  -coordinate system).

ii. A point on a Cartesian coordinate system

iii. A region on a Cartesian coordinate system

From section 1.1.2, remember that the set of ordered pairs ( , ) of real numbers such that  is in the domain of the relation and  is in the range of the relation is represented by the set of points in the  -coordinate.

**Example 1**

Sketch the graph of the relation R if R is the set of ordered pairs ( , ) of real numbers  and  such that  = .

**Solution:**

We take the values of  , calculate the corresponding values of  , plot the resulting points ( , ) and connect the points.

Table 1.4

| | | | | | |
|---|---|---|---|---|---|
| | -2 | -1 | 0 | 1 | 2 |
| | -2 | -1 | 0 | 1 | 2 |

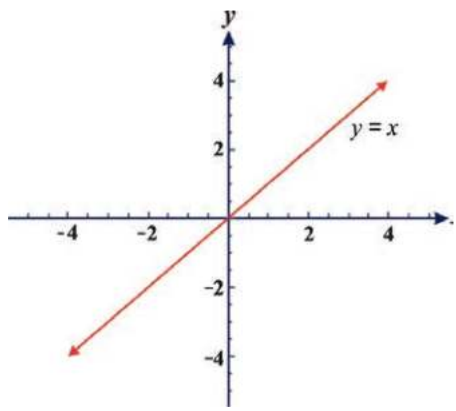



Figure 1.6 the graph of  =

**In general, to sketch graphs of relations involving inequalities, do the following steps**

1. Draw the graph of the line(s) in the relation on the -coordinate system.
2. If the relating inequality is $\leq$ *or* $\geq$, use a solid line; if it is $<$ *or* $>$, use a broken line.
3. Then take arbitrary ordered pairs represented by the points.
4. The region that contains these points representing the ordered pair satisfying the relation will be the graph of the relation.

### Example 2

Sketch the graph of the relation R if R be the set of ordered pairs ( , ) of real numbers and such that $>$ .

**Solution:**

To sketch the graph

1. Draw the line $=$ .

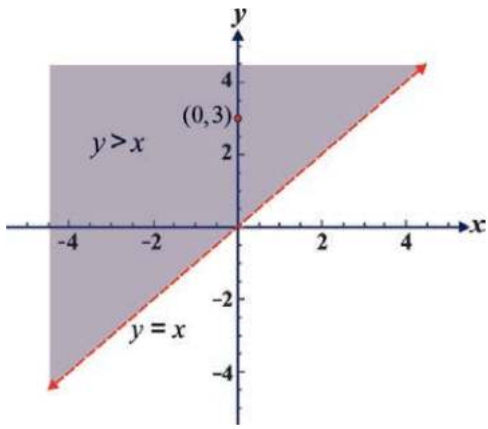



Figure 1.7 the graph of the relation $>$ .

2. Since the relation involves $>$ , use the broken line: "the points on the line $=$ are not included."
3. Take points representing ordered pairs, say $(0, 3)$ and $(0, -2)$ from above and below the line $=$ .
4. The ordered pair $(0, 3)$ satisfies the relation. Hence, points above the line $=$ are members of the relation $R$.

### Example 3

Sketch the graph of the relation $R$ if $R$ be the set of ordered pairs ( , ) of real numbers and such that $\leq$ $+ 2$.

**Solution:**

1. Draw the line $= \quad + 2$.
2. Since the relating inequality is ≤ use the solid line.
3. Select two points representing ordered pairs one from one side and another from the other side of the line. For example, points with coordinates $(0, 4)$ and $(1, 0)$. Obviously, (1,0) satisfies the relation *R* is $\leq \quad + 2$, as $0 \leq 1 + 2$.

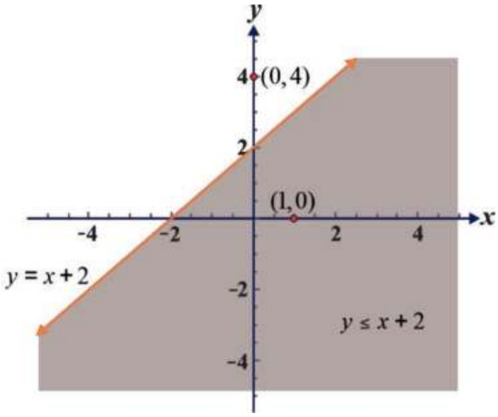



Figure 1.8 the graph of the relation $\leq \quad + 2$.

4. Shade the region below the line, $= \quad + 2$ which contains the point$(1, 0)$. Hence, the graph is the shaded region in figure1.8.

## Exercise 1.6

For each of the following relations, sketch the graph.

a. The relation *R* is set of ordered pairs $( \; , \; ): \; < \; .$
b. The relation *R* is set of ordered pairs $( \; , \; ): \; < - \quad + 7.$
c. The relation *R* is set of ordered pairs $( \; , \; ): \; \leq 3 \quad - 4.$
d. The relation *R* is set of ordered pairs $( \; , \; ): \; \geq 2 \quad + 5.$

### Example 4

Sketch the graph of the relation *R* and determine its domain if *R* be the set of ordered pairs $( \; , \; )$ of real numbers and such that $<$ and $> - \quad + 2$.

**Solution:**

Sketch the graphs of $<$ and $> - \quad + 2$ on the same coordinate system. The two regions have some overlap. The intersection of the two regions is the graph of the relation. So, taking only the common region, we obtain the graph of the relation as shown in figure 1.9. To obtain the intersection point solve the equations $=$ and $= - \; + 2$.

Note that these two lines divide the coordinate system into four regions. Take any points one from each region and check if they satisfy the relation. Say $(2, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.

The point $(2, 0)$ satisfies both inequalities of the relation. So, the graph of the relation is the region that contains $(2, 0)$. Hence, Domain of the relation is the set of real number $:\; > 1$ and Range of $R$ is the set of all real numbers .

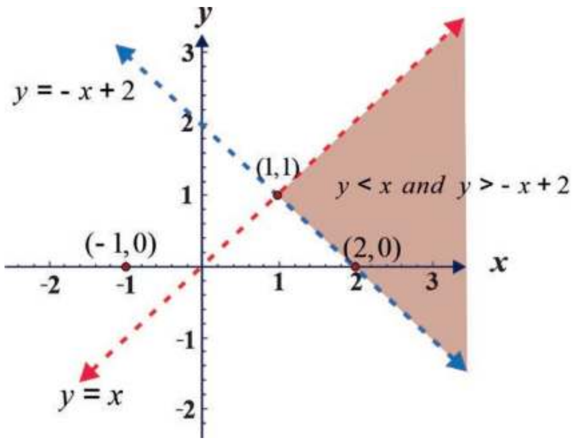



Figure 1.9 the graph of the relation $R$

## Example 5

From the graph of each of the following relations, represented by the shaded region, specify the relation and determine the domain and the range:

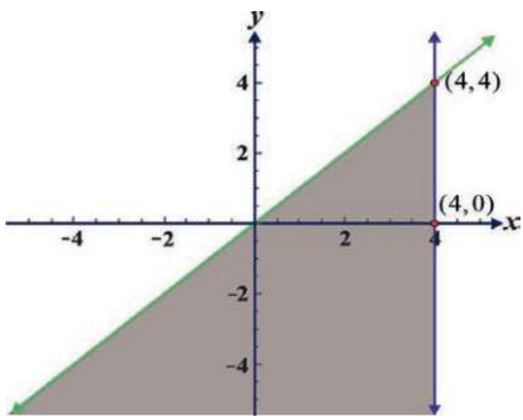



Figure 1.10

**Solution:**

$R = \{( \; , \; ): \; \leq \;$ and $\; \leq 4\};$

Domain $= \{ \; : \; \leq 4\}$ and

Range $= \{ \; : \; \leq 4\}.$

**Note**

A graph of a relation when the relating phrase is an inequality is a region on the coordinate system.

## Exercise 1.7

1. Sketch the graph of the following relations.
   a. *R* is set of ordered pairs ( , ): ≥ + 2 and < − .
   b. *R* is set of ordered pairs ( , ): ≥ 2 and < − .
   c. *R* is set of ordered pairs ( , ): < − − 2 and ≤ −2.
2. From the graph of the following relation, represented by the shaded region, specify the relation and determine its domain and range.

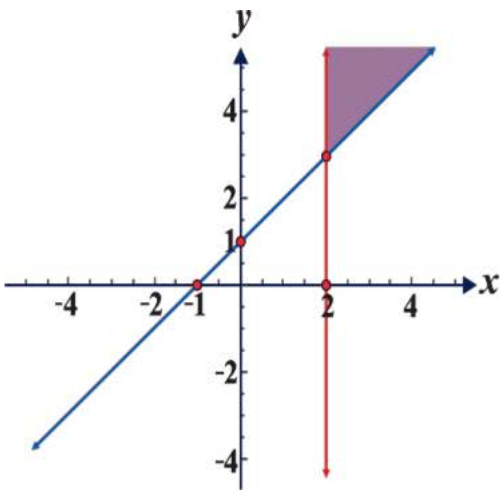



Figure1.11

# 1.2 Functions

In this section, you shall learn about special types of relations which are called functions, the domain and range of function, and combination of functions. The concept of function is the most important point in mathematics. There are terms such as 'map' or 'mapping' used to denote a function.

## 1.2.1 The notion of function

### Activity 1.5

Consider the following relations:

$R_1$ is the set of ordered pairs ( , ) of real numbers and such that $(2,3), (4,5), (3,6), (6,7)$ and $(5,8)$ are members of the relation.

$R_2$ is the set of ordered pairs ( , ) of real numbers and y such that $(2,3), (4,3)$ ,$(3,6), (7,6)$ and $(5,8)$ are members of of the relation.

$R_3$ is the set of ordered pairs ( , ) of real numbers and such that $(2,3), (2,5), (3,6), (3,7)$ and $(5,8)$ are members of the relation.

a. Construct the arrow diagram.

b. How the first elements of the ordered pair are related with the second elements of the ordered pair?

c. In each relation, are there ordered pairs with the same first coordinate?

### Definition 1.3

A function $f$ is a set of ordered pairs with the property that whenever ( , ) and ( $, z$) belong to $f$, then $= z$.

Or

It is a relation in which no two distinct ordered pairs have the same first element.

### Example 1

Consider the following relation $R_1$. $R_1$ is the set of ordered pairs ( , ) of real numbers and such that $(4,5), (6,6), (3,1), (9,7)$ and $(5,2)$ are members of the relation. This relation is a function because no two distinct ordered pairs have the same first element.

## Example 2

Consider the relation $R$ which is the set of ordered pairs ( , ) of real numbers and such that $(2,3), (5,3), (3,6), (7,6)$ and $(5,8)$ are members of the set. Since $(5,3)$ and $(5,8)$ belong to the relation $R$ and $3 \neq 8$ the relation R is not a function.

## Example 3

Look at table 1.5

Table 1.5

| | 1 | 1 | 3 | 3 | 7 |
|---|---|---|---|---|---|
| y | 5 | 5 | −8 | −8 | 4 |

Is R which has members of the ordered pairs $(1,5), (1,5), (3,-8), (3,-8)$ and $(7,4)$ a function?

**Solution:**

Those -values are repeated still it is a function because they are associated with the same value of . Here the ordered pairs $(1,5)$ and $(3,-8)$ are written twice. We can rewrite it by taking a single copy of the repeated ordered pairs. So " $R$" is a function.

## Example 4

Consider the following arrow diagrams in figure1.12

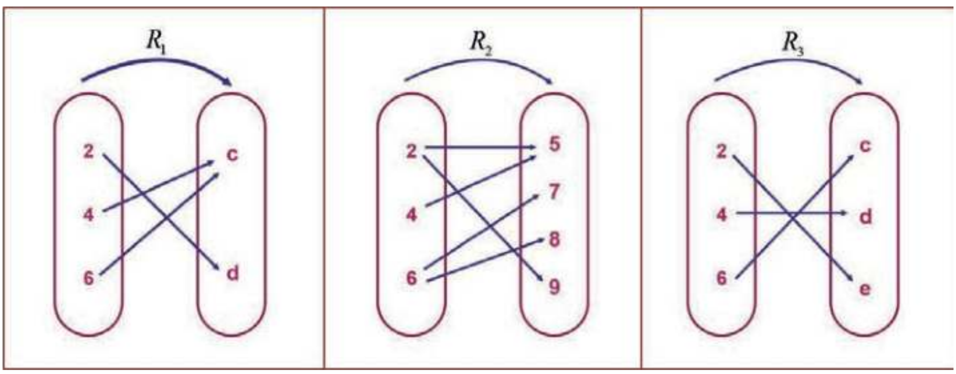



Figure 1.12

Which of the relations $R_1, R_2$ and $R_3$ represent as a function of ?

### Solution:

$R_1$ and $R_3$ are functions, but $R_2$ is not a function because 2 and 6 are both mapped onto two numbers.

**Note**

$f, g$ and $h$ are the most commonly used letters to represent a function; however, any letter of the alphabet can be used.

## Exercise 1.8

Determine whether each of the following relation is a function or not, and give reasons for those that are not functions.

a. $R$ is a set of ordered pairs ( , ) which contains $(6,7), (1,9), (-1,7), (0,0), (4,-4)$.

b. $R$ is a set of ordered pairs ( , ) which contains $(-3,7), (-5,9), (-1,4), (2,0), (-5,3)$

c. The relation $R$ is a set of ordered pairs ( , ): y is a multiple of .

d. The relation $R$ is a set of an ordered pairs ( , ): $y^2 =$ .

e. $R$ is a set of ordered pair $(x, y)$: $y$ is the area of triangle $x$.

## Domain, codomain and range of a function

### Activity 1.6

Consider the following arrow diagram of a function $f$ and find the algebraic rule for $f$.

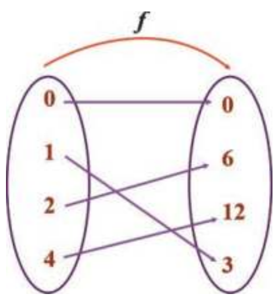



Figure 1.13

The domain of a function $f$ is the set of all values of  for which $f$ is defined and this corresponds to all of the -values on the graph in the -plane. Domain – The set of all possible values which qualify as inputs to a function is known as the domain of the function. In other words, the domain of a function can be defined as the entire set of values possible for independent variables. Co-Domain – The set of all the outputs of a function is known as the range of the function or after substituting the domain, the entire set of all values possible as outcomes of the dependent variable. The range of the function $f$ is the set of all values $f(\ )$ which corresponds to the values on the graph in the -plane.

**Notation:** If is an element in the domain of a function $f$, then the element in the range that is associated with is denoted by $f(\ )$. This is called an image of under the function $f$. The notation $f(\ )$ is referred to function value and we read as '$f\ of$ ' and is the pre-image of $f(\ )$. We can define a function $f(\ ) = 2$ with a domain and codomain of integers. But by thinking about it we can see that the range is just the even integers. The range is a sub-set of the co-domain.

## Example 1

For each of the following functions, find the domain, co-domain and the range.

**a.** $F$ is a set of ordered pairs such that $(3,-2), (5,4), (1,2)$ and $(-3,7)$ are members of $F$

**b.** $R$ is a set of ordered pairs $(\ ,\ )$ of real numbers such that $=$ .

### Solution:

**a.** Domain of $F$ is a collection of $-3, 1, 3$ and $5$, Co-domain of F is a collection of $-2, 2, 4$ and $7$, and the range of $F$ is a collection of $-2, 2, 4$ and $7$.

**b.** Domain of $R$ is a set containing all real numbers and the co-domain and range is the set containing all real numbers .

A function from $A\ to\ B$ can sometimes be denoted as $f: A \to B$, where the domain of the function $f$ is $A$ and the range of the function $f$ is contained in $B$, in which we say $B$ contains the image of the elements of $A$ under the function of $f$.

### Example 2

Consider $f(\ ) = 3\ \ + 4$. Find the domain and the range of the function $f$.

**Solution:**

Since $f(\ ) = 3\ \ + 4$ is defined for every real number , the domain of the function is the set of all real numbers. The range is also the set of real numbers since every real number has a pre-image of a real number such that $= f(\ ) = 3\ \ + 4$.

## Exercise 1.9

1. Find the domain and the range of $F$ where $F$ is a set of ordered pairs such that $(2, -1), (0, 0), (-4, 2)$ and $(-5, 3)$ are members.
2. For each of the following functions, find the domain and range.

   a. $f(\ ) = \ \ + 1$  b. $f(\ ) = 2\ \ - 3$

   c. $f(\ ) = \ ^2$  d. $f(\ ) = \ ^2 + 1$

### Example 1

If $f(\ ) = \sqrt{\ \ - 2}$, then find the domain and range of $f$.

**Solution:**

Since the expression in the radical must be non-zero, $\ - 2 \geq 0$. This implies $\ \geq 2$. Hence, the domain of the function is the set of real numbers greater than or equal to 2. The range of the function is greater than or equal to zero.

**Remark:** If $f: A \to B$ is a function, then, for any included in $A$ (the first coordinate), the image of under $f$, $f(\ )$ is called the functional value *of f at* .

For example, if $f(\ ) = \ \ + 5$, then $f$ at $\ = 4$ is $f(4) = 4 + 5 = 9$.

Finding the functional value of $f$ at is called evaluating the function at .

### Example 2

For the function $f(\ ) = 1 - 3\ $,

**a.** determine the domain and range of $f$.

**b.** find the value of $f(2)$and $f(-1)$.

**Solution:**

**a.** The function $f(\ ) = 1 - 3$ is defined for every real number ; the domain and range of the function $f$ is the set all of real numbers.

**b.** $f(2) = 1 - 3(2) = 1 - 6 = -5$ and $f(-1) = 1 - 3(-1) = 1 + 3 = 4$.

## Exercise 1.10

1. For the function $f(\ ) = - \ + 4$
   a. Determine the domain and the range of $f$.
   b. Find the value of $f(4)$ and $f(9)$.
2. For the function $f(\ ) = \sqrt{\ }$
   a. Determine the domain and the range of $f$.
   b. Find the value of $f(4)$ and $f(9)$.
3. Find the domain and range of each of the following functions.
   a. $f(\ ) = 1 - \ ^2$
   b. $f(\ ) = |\ | + 1$
   c. $f(\ ) = \sqrt{2 - \ }$
4. If $f(\ ) = 2\ + \sqrt{4 - \ }$, then evaluate
   a. $f(-5)$ b. $f(2)$

## 1.2.2 Combination of functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to create a new function.

### Activity 1.7

Consider the functions $f(\ ) = 2\ - 3$ and $g(\ ) = -1$. Then,

a. find $f + g$; $f - g$; $f \cdot g$ and $\frac{f}{g}$.

b. determine the domain and range of $f$ and $g$.

## A. Sum of functions

Let $f$ and $g$ be two functions with overlapping domains. Then, for all   common to both domains, the sum of $f$ and $g$ defined as follows:

$$(f+g)(\ ) = f(\ ) + g(\ )$$

### Example 1

Given $f(\ ) = 2\ \ + 4$ and $g(\ ) = 3\ \ - 1$. Find $(f+g)(\ )$ and evaluate the sum when $\ \ = 2$.

**Solution:**

We know from above the sum of functions $f$ and $g$ is

$$(f+g)(\ ) = f(\ ) + g(\ )$$
$$= (2\ \ + 4) + (3\ \ - 1)$$
$$= 5\ \ + 3$$

and, $(f+g)(2) = 5(2) + 3$

$$= 10 + 3 = 13$$

## B. Difference of functions

Let $f$ and $g$ be two functions with overlapping domains. Then, for all   common to both domains the difference of $f$ and $g$ is defined as follows:

$$(f-g)(\ ) = f(\ ) - g(\ ).$$

### Example 2

Given $f(\ ) = 2\ \ + 1$ and $g(\ ) = \ ^2 + 2\ \ - 1$. Find $(f-g)(\ )$ and evaluate the difference when $\ \ = 2$.

**Solution:**

The difference of the function $f$ and $g$ is

$$(f-g)(\ ) = f(\ ) - g(\ )$$
$$= (2\ \ + 1) - (\ ^2 + 2\ \ - 1)$$

$$= - \ {}^{2} + 2.$$

$$\text{and, } (f-g)(2) = -2^2 + 2$$
$$= -4 + 2 = -2.$$

## Exercise 1.11

1. Given $f(\ ) = 3\ \ + 3$ and $g(\ ) = \ \ - 1$.
   a. Find $(f+g)(\ )$ and evaluate the sum when $\ \ = 2$.
   b. Find $(f-g)(\ )$ and evaluate the difference when $\ \ = 2$.

2. Let $f(\ ) = 2\ \ - 5$ and $g(\ ) = 4\ \ + 1$. Then, evaluate
   a. $(f+g)(\ )$
   b. $(f-g)(\ )$

## C. Product of functions

Let $f$ and $g$ be two functions with overlapping domains. Then, for all common to both domains the product of $f$ and $g$ is defined as follows:

$$(f \cdot g)(\ ) = f(\ ) \cdot g(\ )$$

### Example 1

Given: $f(\ ) = \ {}^{2}$ and $g(\ ) = \ \ - 3$, find $(f \cdot g)(\ )$ and then evaluate the product when $\ \ = 4$.

**Solution:**

$$(f \cdot g)(\ ) = f(\ ) \cdot g(\ )$$
$$= \ {}^{2}(\ \ - 3)$$
$$= \ {}^{3} - 3\ {}^{2}.$$

If $\ \ = 4$, the value of this product is $(f \cdot g)(4) = 4^3 - 3(4^2)$

$$= 64 - 48 = 16.$$

## D. Quotient of functions

Let $f$ and $g$ be two functions with overlapping domains. Then, for all common to both domains the quotient of $f$ and $g$ is defined as follows:

$$\frac{f}{g}(\ ) = \frac{f(\ )}{g(\ )},\ g(\ ) \neq 0.$$

### Example 2

If $f(\ ) = 4$ and $g(\ ) = \quad - 3$, then the quotient of $f$ and $g$ is

$$\left(\frac{f}{g}\right)(\ ) = \frac{f(\ )}{g(\ )} = \frac{4}{-3}.$$

## Exercise 1.12

Given $f(\ ) = \quad - 1$ and $g(\ ) = \ ^2$,

a. Find$(f \cdot g)(\ )$ and then evaluate the product when $\quad = 3$.

b. Find $\left(\frac{f}{g}\right)(\ )$ and then evaluate the quotient when $\quad = 3$.

c. Find $\left(\frac{g}{f}\right)(\ )$ and then evaluate the quotient when $\quad = 3$.

## The domain of combination of functions

### Activity 1.8

Consider the functions $f(\ ) = \quad - 3$ and $g(\ ) = \quad + 1$. Then,

a. find $f + g$; $f - g$; $f \cdot g$ and $\frac{f}{g}$.

b. determine the domain and the range of each function $f$ and $g$.

c. is the domain of $f$ and $g$ the same as the domain of $f + g$? Why?

The domain of an arithmetic combination of functions f and $g$ ($f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$) consists of real numbers common to the domain of $f$ and $g$. In the case of quotient of functions $\frac{f(x)}{g(x)}$, there is further restriction that $g(\ )$ is not equal to zero.

### Example 1

Given $f(\ ) = \quad - 5$ and $g(\ ) = 2\quad + 1$. Find the domain of$(f + g)(\ )$ and $(f - g)(\ )$.

**Solution:**

The domain $f$ and $g$ is the set of all real numbers. So, the domain of $f + g$ and $f - g$ is the set of all real numbers.

**Example 2**

Given: $f(\ ) = 2$ and $g(\ ) = \ ^2$, find $(f \cdot g)(\ )$ and $\frac{f(x)}{g(x)}$

**Solution:**

The domain $f$ and $g$ is the set of all real numbers. So, the domain of $f \cdot g$ and $\frac{f}{g}$ is the set of all real numbers. Here, $g$ cannot be zero. So, the domain of $\frac{f}{g}$ is $\mathbb{R}/\{0\}$ (the set of real numbers without zero) since $g$ cannot be zero.

**Example 3**

Find $\frac{f}{g}(\ )$and $\frac{g}{f}(\ )$ for functions given by $f(\ ) = \sqrt{\ }$ and $g(\ ) = \sqrt{4 - \ ^2}$. Then, find the domain of $\frac{f}{g}$ and $\frac{g}{f}$.

**Solution:**

The quotient of $f$ and $g$ is given by $\left(\frac{f}{g}\right)(\ ) = \frac{f(\ )}{g(\ )} = \frac{\sqrt{\ }}{\sqrt{4- \ ^2}}$ and the quotient of $g$ and $f$ is $\left(\frac{g}{f}\right)(\ ) = \frac{g(\ )}{f(\ )} = \frac{\sqrt{4- \ ^2}}{\sqrt{\ }}$.

The domain of $f$is $[0, \infty)$and the domain of $g$ *is* $[-2, 2]$. The intersection of these domains is $[0, 2]$. Here, $g$ cannot be zero. So, the domain of $\frac{f}{g}$ *is* $[0,2)$ and domain of $\frac{g}{f}$ *is* $(0,2]$ since $f$ cannot be zero.

## Exercise 1.13

1. Let $f(\ ) = 2\ \ - 5$ and $g(\ ) = \ \ + 1$, then evaluate

   a. $(f + g)(\ )$
   b. $(f - g)(\ )$
   c. $(fg)(\ )$
   d. $\left(\frac{f}{g}\right)(\ )$
   e. the domain of $f + g$ , $f - g$ $f \cdot g$ and $\frac{f}{g}$

2. Let $f(\ ) = \ ^2 + 5$ and $g(\ ) = \sqrt{1 - \ }$. Find

   a. $(f + g)(\ )$   b. $(f - g)(\ )$

   c. $(f . g)(\ )$   d. $(\frac{f}{g})(\ )$

   e. the domain of $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$.

## Example 1

Given: $f(\ ) = 7 - 2\ $ and $g(\ ) = -\ - 6$. Determine:

**a.** $3f + g$   **b.** $4g - 3f$

**c.** $(4f)g$   **d.** $\frac{4g}{5f}$

**Solution:**

**a.** $(3f + g)(\ ) = 3f(\ ) + g(\ )$

$= 3(7 - 2\ ) + (-\ - 6)$

$= 21 - 6\ - \ - 6 = 15 - 7\ .$

**b.** $(4g - 3f)(\ ) = 4g(\ ) - 3f(\ ) = 4(-\ - 6) - 3(7 - 2\ )$

$= -4\ - 24 - 21 + 6\ = 2\ - 45.$

**c.** $[(4f) \cdot g](\ ) = 4f(\ ) \cdot g(\ )$

$= 4(7 - 2\ )(-\ - 6)$

$= 4(-7\ - 42 + 2\ ^2 + 12\ )$

$= 4(2\ ^2 + 5\ - 42) = 8\ ^2 + 20\ - 168.$

**d.** $\frac{4g}{5f} = \frac{4(-x-6)}{5(7-2x)} = \frac{-4x-24}{35-10x} = \frac{4x+24}{10x-35}$.

## Exercise 1.14

1. Given: $f(\ ) = 2 -\ $ and $g(\ ) = -2\ + 3$.

   a. Determine $f + 2g$ and evaluate $(f + 2g)(2)$.

   b. Determine $2f - g$ and evaluate $(2f - g)(2)$.

   c. Determine $2f \cdot g$ and evaluate $(2f \cdot g)(2)$.

   d. Determine $\frac{3f}{2g}$ and evaluate $\frac{3f}{2g}(2)$

2. If f and g are any functions, is the domain of $f+g,\ f-g,\ f\cdot g$ and $\frac{f}{g}$ differ from the domain of $f$ and $g$? If your answer is yes, why?

**Note**

A numerical relation $R$ is a function if and only if no **vertical line** in the plane intersects the graph of $R$ in more than one point.

**Vertical Line Test:**

Vertical line test is used to determine whether a graph of a curve is a function or not. If any curve cuts a vertical line at more than one point then the curve is not a function.

**Example 1**

The figure1.14 describes the graph is not a function.

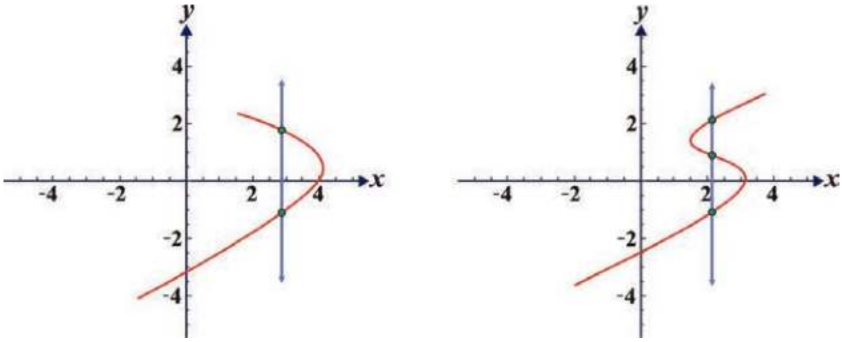



Figure 1.14

## 1.2.3 Types of functions

There are different types of function. These are

- One-to-one function (Injective function)
- Onto function (Surjective function)
- One-to-one correspondence (Bijective)

### One-to-one function (Injective function)

If each element in the domain of a function has a distinct image in the co-domain, then the function is said to be one -to- one function.

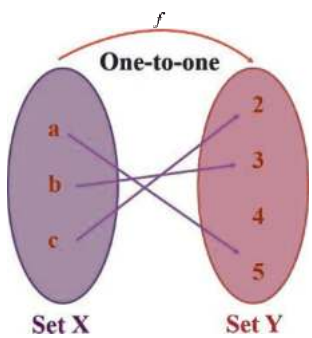



Figure 1.15

## Definition 1.4

A function $f: A \to B$ is called one-to-one if and only if for all $_1, \ _2 \in A$,

$f(\ _1) = f(\ _2)$ implies $_1 = \ _2$.

### Example 1

Let $f: R \to R$ given by $f(\ ) = 3\ \ + 5$. Show that $f$ is one-to-one.

**Solution:**

Let $_1, \ _2 \in R, f(\ _1) = f(\ _2)$ implies $3\ _1 + 5 = 3\ _2 + 5$ implies $_1 = \ _2$

Therefore, $f$ is one-to-one.

### Example 2

In figure 1.16, which of the following sets of values represent a one-to-one function?

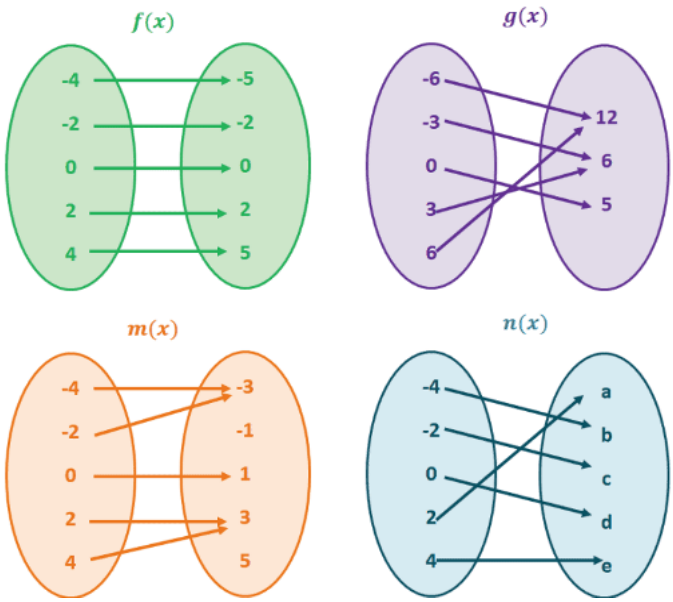



Figure 1.16

**Solution:**

For the first set $f(\ )$, we can see that each element from the right side is paired up with a unique element from the left. Hence, $f(\ )$ is a one-to-one function.

The set $g(\ )$, shows a different number of elements on each side. This alone will tell us that the function is not a one-to-one function.

Some values from the left side correspond to the same element found on the right, so $m(\ )$ is not a one-to-one function as well.

Each of the elements on the first set corresponds to a unique element on the next, so $n(\ )$ represents a one-to-one function.

## Onto function (Surjective Function)

A function is called an onto function if each element in the co-domain has at least one pre-image in the domain (see figure 1.17).

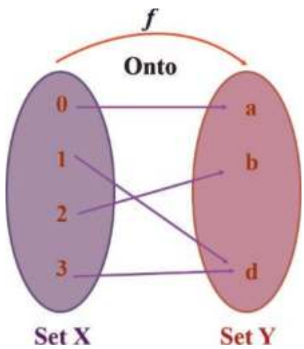



Figure 1.17

## One-to-one correspondence (Bijective function)

A function $f: R \rightarrow R$ is said to be a one-to-one correspondence if $f$ is both one-to-one and onto.

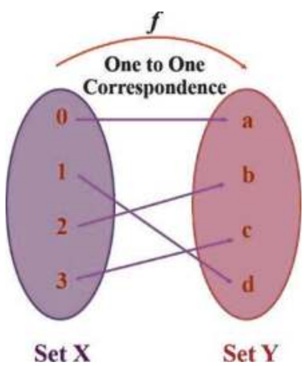



Figure 1.18

## Example 3

Which of the following sets of values represents an injective and surjective function?

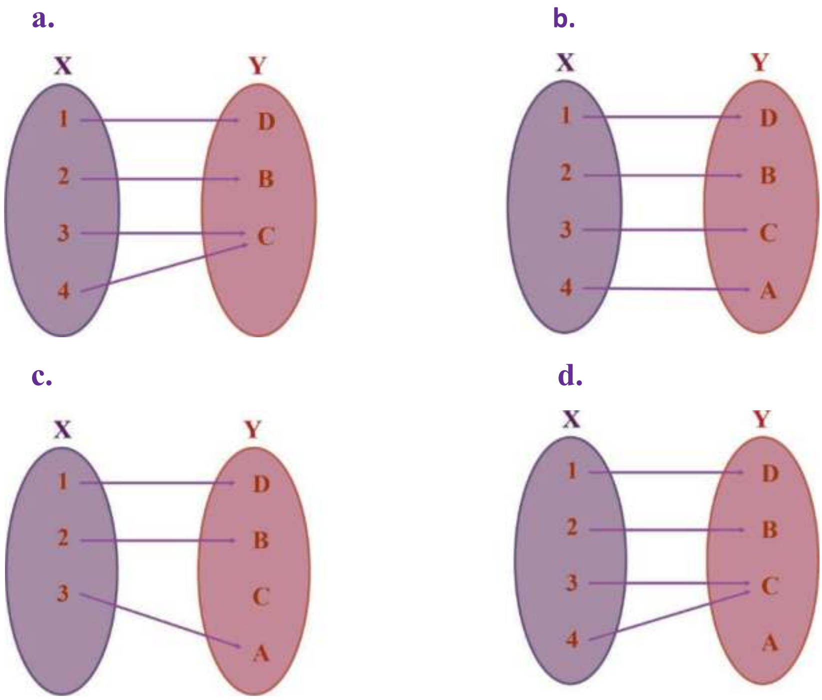



Figure 1.19

**Solution:**

**a.** Not one-to-one but it is an onto (**surjective)** function.

**b.** Both one-to-one and onto function.

**c.** An injective non-surjective function (injection).

**d.** A non-injective non-surjective function.

## Example 4

The function $f(\ ) = \ ^2$ from the set of positive real numbers to the set of positive real numbers is both injective and surjective. Thus, it is also a bijective. Is it true that whenever $f(\ ) = f(\ )$, $=$ ? Imagine $= 3$, then $f(\ ) = 9$. Now I say that $f(\ ) = 9$, what is the value of ? It is 3, so $=$ .

## Exercise 1.15

1. Which of the following mappings are injective, surjective, bijective function?

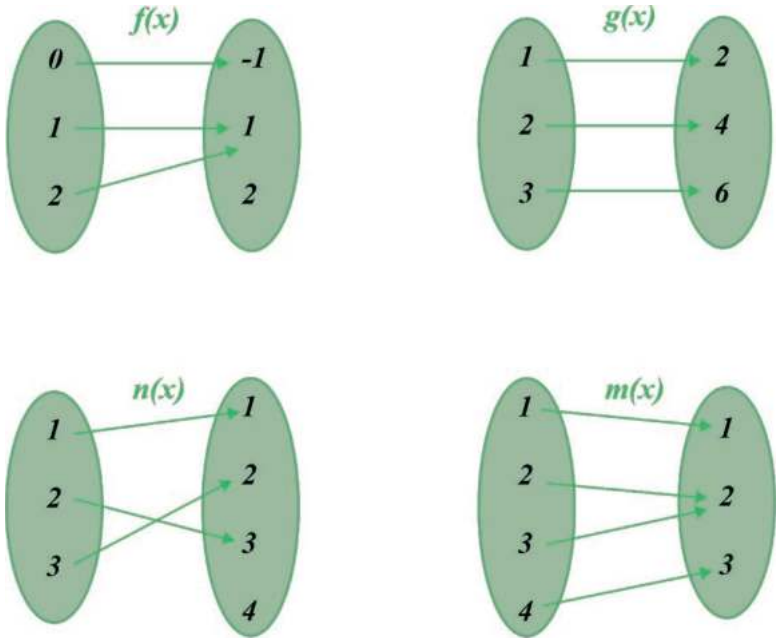



Figure 1.20

2. Which of the following are one-to-one functions?

   a. $A$ is the set of ordered pair ( , ): $=|\quad - 2|$.

   b. $B = f: R \to R$ given by $f(\ ) =$ .

3. Which of the following functions are onto?

   a. $f: R \to R, f(\ ) = 2 \quad - 3$

   b. $g: [0, \infty) \to \mathfrak{R}, g(\ ) = \ ^2$

4. Identify if the following function is an injective, surjective, and/or bijective function?

   a. $f: R \to R, f(\ ) = \quad + 1$  b. $f: R \to R, f(\ ) = 2$

## 1.2.4 Graphs of functions

In this section, you will learn how to draw graphs of functions such as $= ax + b$ **and** $= a\ ^2 + bx + c$ with special emphasis on linear and quadratic functions. You will study some of the important properties of graphs as in the following:

### Definition 1.5

If $f$ is a function with domain $A$, then the graph of $f$ is the set of all ordered pairs $\{(\ ,f(\ )),\ \ \in A\}$

That is, the graph of $f$ is the set of all points $(\ ,\ )$ such that $f(\ ) =\ $. This is the same as the graph of the equation $f(\ ) =\ $ discussed in Cartesian coordinates. The graph of a function allows us to translate between algebra and pictures or geometry.

## Graph of linear function:

### Definition 1.6

If $a$ and $b$ are fixed real numbers, $a \neq 0$, then $f(\ ) = ax + b$ for every real number  is called a linear function. If $a = 0$, then $f(\ ) = b$ is called a constant function. Sometimes linear functions are written in the form $\ = ax + b$.

### Example 1

$f(\ ) = 2\ \ + 1$ is a linear function with $a = 2$ and $b = 1$

### Example 2

$f(\ ) = 2$ is a constant function.

To work on graphic functions, you can pick a few values of  and calculate the corresponding values of  or $f(\ )$, plot the resulting points $(\ ,f(\ ))$ and connect the points.

### Example 3

Draw the graph of $f(\ ) = 2$

**Solution:**

Construct a table for the value of the function; plot the ordered pairs and draw a line through the points to get the required graph.

Table 1.6

| | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(\ )$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

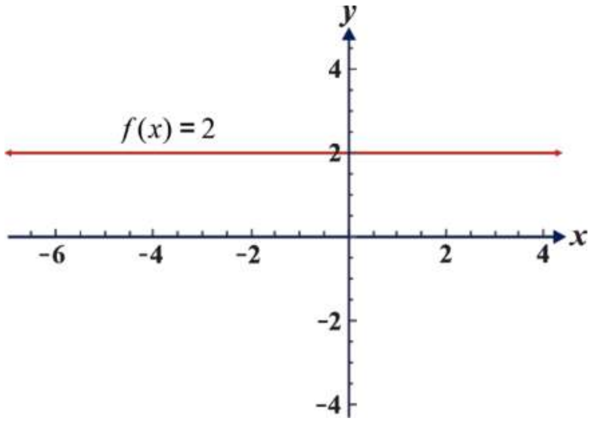



Figure 1.21 Graph of the constant function $f(\ ) = 2$

## Example 4

Fill in the tables shown in Table 1.7 for the function $f(\ ) =$ and draw its graph.

Table 1.7

| | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(\ )$ | | | | | | | |

If $f(-3) = -3, f(-2) = -2, f(-1) = -1, f(0) = 0, f(1) = 1,\ f(2) = 2,$ $f(3) = 3.$

So, the table becomes

Table 1.8

| | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(\ )$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

When you plot the corresponding points on the Cartesian plane and connect the points to get a picture of the graph of function, the ordered pairs will give you a graph in the shape shown in Figure 1.22. The domain is the set of all real number and the range is the set of all real numbers .

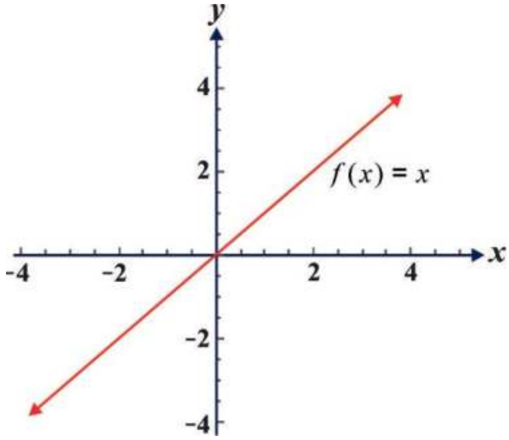



Figure 1.22 the graph of $f( ) =$

## Example 5

Consider the linear function $f( ) = \quad + 3$ and evaluate the values of the function for the values in Table 1.9 and draw its graph.

Table 1.9

| | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|---|
| $f( ) = \quad + 3$ | | | | | | | |

If $\quad = -4$, $f(-4) = -1, f(-3) = 0$, $f(-2) = 1, f(-1) = 2$, $f(0) = 3$, $f(1) = 4$, $f(2) = 5$, the table becomes

Table 1.10

| | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|---|
| $f( ) = \quad + 3$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |

Table 1.10 is pairing the values of and $f( )$. This is taken as a representation of set of ordered pairs $(-4, -1), (-3, 0), (-2, 1), (-1, 2), (0, 3), (1, 4)$ and $(2, 5)$. Now you can plot these points in a coordinate system to draw the graph of the given function.

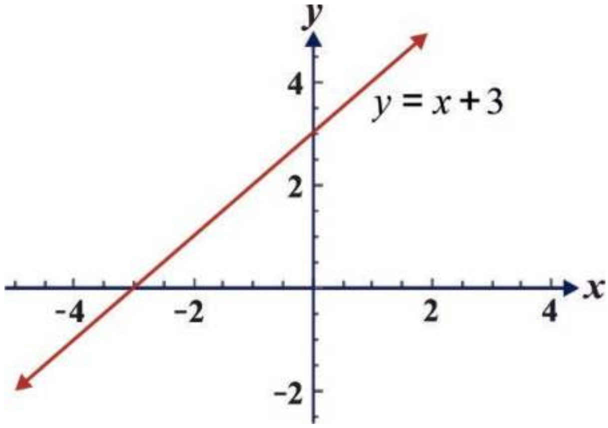



Figure 1.23 the graph of $f(\ ) = \ + 3$

## Example 6

Draw the graph of the linear function $f(\ ) = -2\ \ + 6$.

Table 1.11

| | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $f(\ ) = -2\ \ + 6$ | 10 | 8 | 6 | 4 | 2 | 0 |

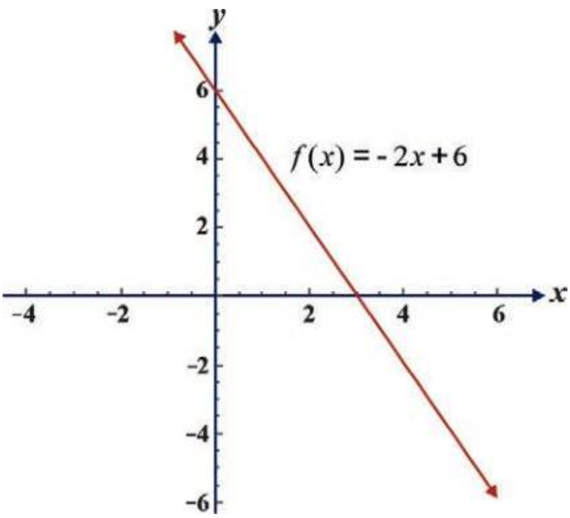



Figure 1.24 the graph of $f(\ ) = -2\ \ + 6$

## Exercise 1.16

Construct tables of values of the following functions for the given domains:

a. $f( \ ) = 4 \quad + 1; \qquad = -3, -2, -1, 0, 1, 2, 3$

b. $f( \ ) = -2 \quad + 5; \qquad = -1, -0.5, 0, 2, 3, 4$

c. $f( \ ) = 7 - 3 \ ; \qquad = -1, 0, 1, 2, 3, 4$

d. $f( \ ) = \frac{x}{4} + 1; \qquad = -8, -4, -2, 0, 2, 4$

## Example 1

Draw the graph of the following functions in the same Cartesian coordinate system.

**a.** $f( \ ) = \quad + 2$ **b.** $g( \ ) = - \quad + 2$

**c.** $h( \ ) = 2 \quad + 2$ **d.** $k( \ ) = -2 \quad + 2$

**Solution:**

We need two points to draw a line. However, we generally choose three, and the third point is a good check that we do **not** make a mistake.

**a.** $f( \ ) = \quad + 2: f(0) = 2,$
$f(1) = 3, f(2) = 4,$

**b.** $g( \ ) = - \quad + 2 : g(0) = 2,$
$g(1) = 1, g(2) = 0,$

**c.** $h( \ ) = 2 \quad + 2 : h(0) = 2,$
$h(1) = 4, h(2) = 6,$

**d.** $k( \ ) = -2 \quad + 2 : k(0) = 2,$
$k(1) = 0, k(2) = -2.$

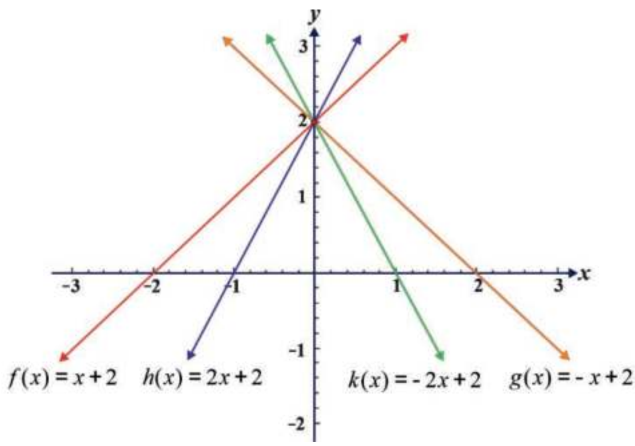



Figure 1.25 the graph of four functions $f, g, h, k$ intersecting at a point (0,2).

As indicated on Fig.1.25, the graphs of the four functions cross the -axis at $= 2$ because the value of b is always 2 for the equation $= ax + b$ where b is the y-intercept. Thus, you can see that if $a > 0$ then the straight line goes up as increases, and the bigger $a$ gets the faster the line goes up. Similarly, if $a < 0$ then the line goes down as  increases, and the bigger $a$ gets in absolute terms, the faster the line goes down.

**From the graphs given above, you have noticed that**:

**a.** Graphs of linear functions are straight lines.

**b.** If $a > 0$, then the graph of the linear function $f(\ ) = ax + b$ is increasing.

**c.** If $a < 0$, then the graph of the linear function $f(\ ) = ax + b$ is decreasing.

**d.** If $a = 0$, then the graph of the linear function $f(\ ) = b$ is a horizontal line.

## Exercise 1.17

Draw the graph of the following functions in the same Cartesian coordinate system.

a. $f(\ ) = 2\ \ + 1$  b. $g(\ ) = 2\ \ - 1$

c. $h(\ ) = -2\ \ + 1$  d. $k(\ ) = -2\ \ - 1$

e. $m(\ ) = 1$  f. $n(\ ) = -1$

## Slope and intercept:

The slope indicates the steepness of a line and the intercept indicates the location where it intersects an axis. Linear functions are written in the form $\ = ax + b$ and $a$ is called its slope and $b$ is its  -intercept.

### Example 1

If $f(\ ) = 3\ \ - 6$, then find the  and  -intercepts.

**Solution:**

To find  - intercept, put $f(\ ) = \ \ = 0$ and then solve for  as:

$$0 = 3\ \ - 6,$$

$$= 2.$$

So, the  - intercept is $(2, 0)$.

To find  -intercept, put $\ = 0$. And solve, you have $f(0) = -6$.

So, the  - intercept *is* $(0, -6)$.

## Example 2

Draw the graph of the function $f(\ ) = 2\ \ - 3$.

**Solution:**

Find the and - intercept.The - intercept is the ordered pair where $= 0$ that is $(\frac{3}{2}, 0)$. And the - intercept is the ordered pair where $= 0$ that is $(0, -3)$. Plot these intercepts on a coordinate system and draw a line that passes through them.

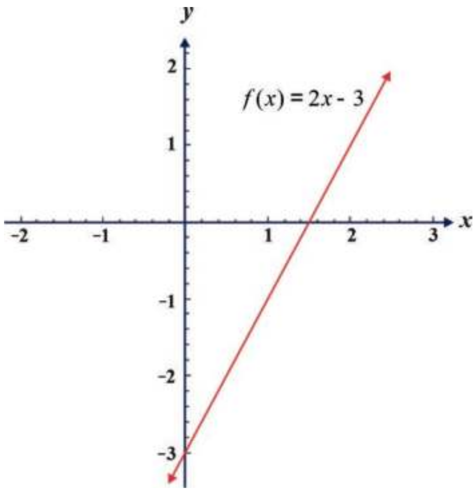



Figure 1.26 the graph of $f(\ ) = 2\ \ - 3$

## Example 3

If $f(\ ) = 2 -\ $, is the graph of the function increasing or decreasing function?

**Solution:**

Since $f(\ ) = 2 -\ $ is the same as $f(\ ) = -\ \ + 2$ and the coefficient of is $-1$, the graph the function is decreasing.

## Exercise 1.18

1. Determine the slope, -intercept and -intercept of the following linear functions:
   a. $+\ \ - 2 = 0$ b. $2\ \ + 2\ \ = 3$
   c. $f(\ ) + 7 = 2$ d. $f(\ ) = -3\ \ - 5$
2. Given the following functions:
   $f(\ ) = 3\ \ - 1, g(\ ) = -\ \ + 2, h(\ ) = -2\ $ and $k(\ ) = 1$.
   a. Which one is a decreasing function?
   b. Find slope and -intercept and -intercept of each function.

3. Sketch the graph of each of the following using intercepts:

 a. $= 2 \quad + 4$  $b.$ $3 + 4 \quad = 5$  $c.$ $= -4 \quad + 3$

## Graphs of quadratic functions

A function defined by $f(\ ) = a \ \ ^2 + bx + c$ where $a, b, c$ are real numbers and $a \neq 0$ is called a quadratic function. The point $a$ is the leading coefficient of $f$.

### Example 1

$f(\ ) = 3 \ \ ^2 - 2 \quad + 5$ is a quadratic function with $a = 3$, $b = -2$ and $c = 5$.

Quadratic functions are useful in many applications in mathematics when a linear function is not sufficient. For example, the motion of an object thrown either upward or downward is modeled by a quadratic function. The graph of a quadratic function is a curve called a parabola. Parabolas may open upward or downward and vary in" width" or" steepness", but they all have the same basic" U" shape. All parabolas are symmetric with respect to a line called the axis of symmetry. A parabola intersects its axis of symmetry at a point called the vertex of the parabola.

Many quadratic functions can be graphed easily by hand using the techniques of stretching/ shrinking and shifting (translation) the parabola $= \ \ ^2$.

**Note**

The simplest form of a quadratic equation is $= \ \ ^2$ when $a = 1$ and $b = c = 0$.

**Activity 1.9**

a. Make a table of ordered pairs that satisfy the function $f(\ ) = \ \ ^2$.

b. Find the  and  intercepts of $f$.

c. Plot the points $(\ ,\ \ ^2)$ on  -coordinate system.

d. Find the domain and range of $f$.

The following are observations of the graph of $h(\ ) = \ ^2$.

- Since squaring any number gives a positive number, the values of are all positive, except when $= 0$, in which case $= 0$.
- As increases in size, so does $\ ^2$ but the increase in the value of $\ ^2$ is 'faster' than the increase in .
- The graph of $= \ ^2$ is symmetric about the -axis ( $= 0$). For example, if $= 3$, the corresponding value is $3^2 = 9. If \quad = -3$, then the value is $(-3)^2 = 9$. The two - values are equidistant from the -axis: one to the left and one to the right, but the two values are in the same height above the - axis.

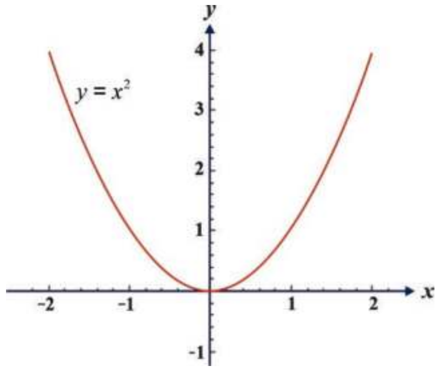



Figure 1.27 the graph of $= \ ^2$

Referring to figure1.27, we observe the following

- the line $= 0$ ( -axis) is called the line of symmetry for this quadratic function.
- the line $= 0$ ( -axis) is called the orthogonal axis for this quadratic function.

If the equation is, say, $= \ ^2$ then the graph will be similar to that of $= 2\ ^2$ but will lie above it. For example, when $= 1$ the value of $\ ^2$ is 1, but the value of $2\ ^2$ is 2. The value for $= 2\ ^2$ is above that of $= \ ^2$. Similarly, for the equation $= \frac{x^2}{2}$, the graph is similar to that of $= \ ^2$ except lying below.

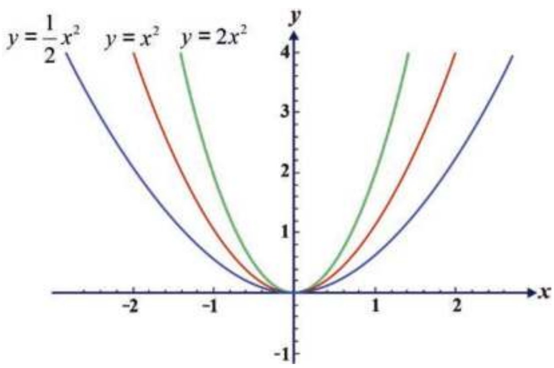



Figure 1.28 the graph of $= a\ ^2$

Sketch of $= a\ ^2$ for different positive values of $a$ are shown in Figure 1.28.

Consider now the choice $a = -1$, with the equation $\quad = - \quad ^2$. In this case the graph of the equation will have the same shape but now, instead of being above the -axis it is below. When $\quad = 1$ the corresponding value is $-1$.

Table 1.12

| | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(\ ) = - \quad ^2$ | $-9$ | $-4$ | $-1$ | $0$ | $-1$ | $-4$ | $-9$ |
| $g(\ ) = -2 \quad ^2$ | $-18$ | $-8$ | $-2$ | $0$ | $-2$ | $-8$ | $-18$ |
| $h(\ ) = -\frac{1}{2} \quad ^2$ | $-\frac{9}{2}$ | $-2$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-2$ | $-\frac{9}{2}$ |

Examples of $\quad = a \quad ^2$ for various negative values of $a$ are sketched in figure 1.29.

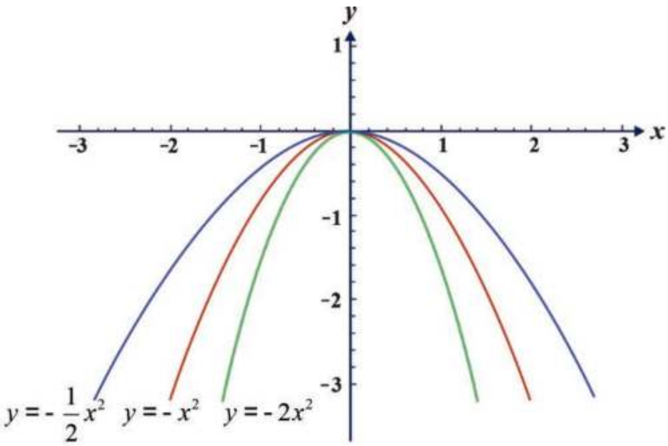



Figure 1.29 the graph of $\quad = a \quad ^2, a = -2, -\frac{1}{2}, -1$

## Exercise 1.19

Draw the graph of the following functions by constructing tables of values:

$\quad = -2, -1, 0, 1, 2.$

a. $f(\ ) = \frac{1}{4} \quad ^2$  b. $g(\ ) = -\frac{1}{4} \quad ^2$

c. $h(\ ) = \frac{3}{2} \quad ^2$  d. $k(\ ) = -\frac{3}{2} \quad ^2$

### Graph of $y = ax^2 + c$.

This type of quadratic function is similar to the basic ones of the previous functions discussed but with a constant c added in the function $= a\ ^2$, i.e., having the general form

$= a\ ^2 + c$. As an example of this, $=\ ^2 + 2$. Comparing this with the function $=\ ^2$, the only difference is the addition of 2 units. When $= 1$,

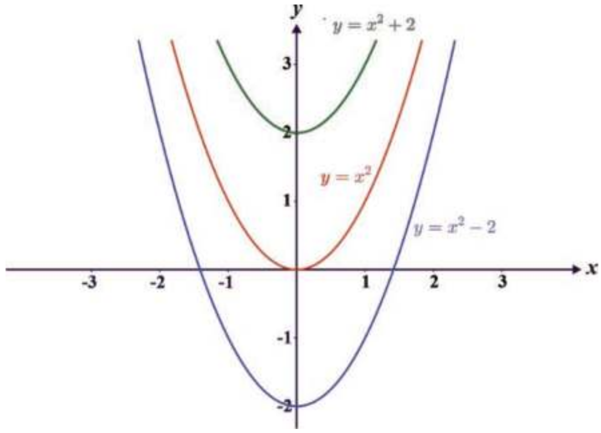



Figure 1.30

$^2 = 1$, but $^2 + 2 = 1 + 2 = 3$. When $= 2$, $^2 = 4$, but $^2 + 2 = 4 + 2 = 6$.

That is the values of $=\ ^2$ have been lifted by 2 units. This happens for all of the values so the shape of the graph is unchanged but it is lifted by 2 units. Similarly, the graph of $=\ ^2 - 2$ will be lowered by 2 units.

### Graph of $y = a(x - k)^2$

In the examples considered so far, the axis of symmetry is the -axis, i.e., the line $= 0$. The next possibility is a quadratic function which has its axis of symmetry not on the -axis.

A case in point to this function: $= (\ - 3)^2$ has the same shape and the same orthogonal axis as $=\ ^2$ but the axis of symmetry is the line $= 3$. i.e., shift to the right by 3 units. The points $= 0$ and $= 6$ are equidistant from 3. When $= 0$ the value is 9. When $= 6$ the value is $(6 - 3)^2 = 9$.

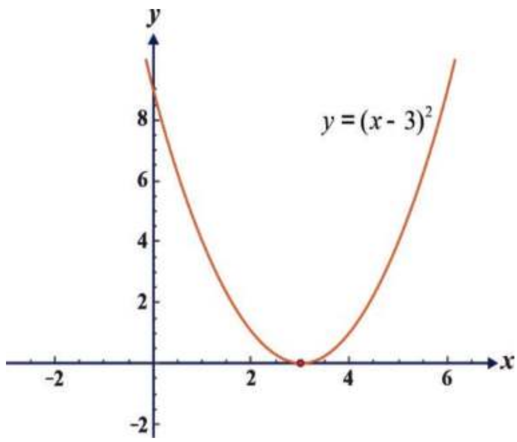



Figure 1.31 the graph of $= (\ - 3)^2$

The points on the curve at these values are both 9 units above the -axis.

This is true for all numbers which are equidistant from 3.

**From the graph of quadratic functions of the form $f(x) = ax^2$, $y = ax^2 + c$, $a \neq 0$, $c$ is any real number, we can summarize:**

1. If $a > 0$, the graph opens upward and if $a < 0$, the graph opens downward.
2. The vertex is $(0,0)$ for $f(\ ) = a\ ^2$ and $(0, c)$ for $\ = a\ ^2 + c$.
3. The domain is all real numbers.
4. The vertical line that passes through the vertex is the axis of the parabola (or the axis of symmetry).
5. If $a > 0$, the range is the set of non- negative real number for $f(\ ) = a\ ^2$ and the set of real number such that $\ \geq c$ for $\ = a\ ^2 + c$
6. If $a < 0$, the range is the set of non- positive real number for $f(\ ) = a\ ^2$ and the set of real number such that $\ \leq c$ for $\ = a\ ^2 + c$

## Exercise 1.20

Draw the graph of the following functions.

a. $f(\ ) = \ ^2 + 1$
b. $g(\ ) = \ ^2 - 3$
c. $h(\ ) = 2\ ^2 + 2$
d. $k(\ ) = (\ - 1)^2$
e. $m(\ ) = (\ + 1)^2$

### Graphs of $\ = a(\ - k)^2 + m\ (a < 0)$.

So far, two separate cases have been discussed: first a standard quadratic function has its orthogonal axis shifted up or down; second a standard quadratic function has its axis of symmetry shifted left or right. The next step is to consider quadratic functions that incorporate all shifts.

### Example 1

The quadratic function $\ = \ ^2$ is shifted so that its axis of symmetry is at $\ = 3$ and its orthogonal axis is at $\ = 2$.

**a.** Write down the equation of the new curve.

**b.** Find the coordinates of the point where it crosses the -axis.

c. Sketch the new curve shifted from $= \quad ^2$

**Solution:**

a. The new curve $= ( \quad - 3)^2$ is symmetric about $\quad = 3$ and is shifted up by 2 units, so its equation is $\quad = ( \quad - 3)^2 + 2$.

b. The curve crosses the  -axis, when $\quad = 0$. Putting this into the equation $= ( \quad - 3)^2 + 2$, the corresponding value of  is

$= (0 - 3)^2 + 2 = 11,$

so, the curve crosses the $\quad -$axis at $\quad = 11$.

c. The curve is shown in figure 1.32.

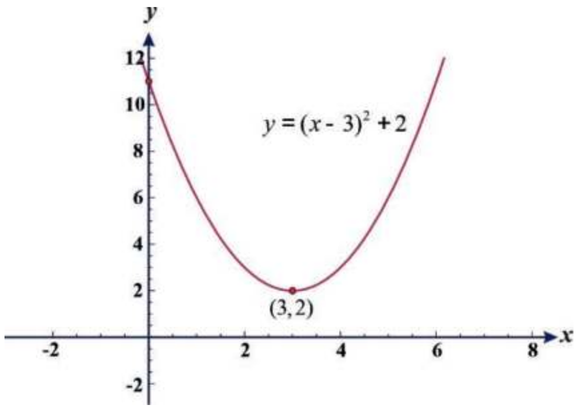



Figure 1.32 the graph of $= ( \quad - 3)^2+2$

Note that the vertex of the graph of $f( \quad ) = ( \quad - k)^2 + c$ is $(k, c)$.

## Exercise 1.21

1. The curve $\quad = -2 \quad ^2$ is shifted so that its axis of symmetry is the line $\quad = -2$ and its orthogonal axis is $\quad = 8$.
   a. Write down the equation of the new curve.
   b. Find the coordinates of the points where this new curve cuts the  and  axes.
   c. Sketch the curve
2. Find the vertex and draw the graph of the following functions.
   a. $f( \quad ) = ( \quad - 1)^2 + 2$
   b. $g( \quad ) = ( \quad - 2)^2 - 3$
   c. $h( \quad ) = ( \quad + 1)^2 + 1$

The final section is about sketching general quadratic functions, i.e., one of the forms

$= a \quad ^2 + bx + c$ .

The algebraic expression must be rearranged so that the line of symmetry and the orthogonal axis can be determined. The procedure required is completing the square.

### Example 1

A quadratic function is given by $\quad = \quad ^2 + 6 \quad + 11$.

**a.** Apply completing the square method, and find the vertex of the function.

**b.** Use this to determine the axis of symmetry and the orthogonal axis of the curve.

**c.** Find the points on the  and  axes where the curve crosses them.

**d.** Sketch the function.

**Solution:**

**a.** Completing the square:

$$= \quad ^2 + 6 \quad + 11$$

$$= \quad ^2 + 6 \quad + 9 - 9 + 11$$

$$= ( \quad + 3)^2 + 2.$$

Thus, the vertex is $(-3, 2)$.

**b.** This is the function $\quad = \quad ^2$ moved to the left so that its axis of symmetry is $\quad = -3$ and shifted up by 2, i.e., its orthogonal axis is $\quad = 2$.

**c.** The function is $\quad = ( \quad + 3)^2 + 2$. It will not cross the *x*-axis, i.e., the graph has not  -intercept. Putting $\quad = 0$ into the original form of the function $\quad = \quad ^2 + 6 \quad + 11$, gives $\quad = 11$, i.e., it crosses the  -axis at $\quad = 11$.

**d.** The function is shown in figure1.33.

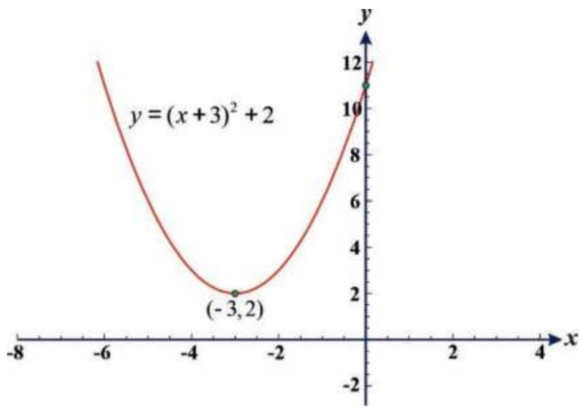



Figure 1.33 the graph of the function $\quad = \quad ^2 + 6 \quad + 11$

## Exercise 1.22

1. A quadratic function is given as $= \quad {}^2 + 4 \quad + 11.$
   - a. completes the square on this function and find the vertex.
   - b. use this to determine the axis of symmetry and the orthogonal axis of the curve.
   - c. finds the points on the  and  axes where the curve crosses them.
   - d. Sketch the function.
2. A quadratic function is given as $= \quad {}^2 - 8 \quad + 14$, and find the vertex.
   - a. completes the square on this function.
   - b. Sketch the function.
3. Sketch each of the following quadratic functions.

   a. $= \quad {}^2 + 2 \quad + 1$ b. $= \quad {}^2 - 6 \quad + 5$ c. $= \quad {}^2 + 2 \quad + 5$

### Graphs of $= a( \quad - k)^2 + m \ (a < 0).$

#### Example 1

A quadratic function is given as $= - \quad {}^2 + 2 \quad + 3.$

**a.** Complete the square on this function.

**b.** Use this to determine the axis of symmetry and the orthogonal axis of the curve.

**c.** Find the points on the *x* and *y* axes where the curve crosses them.

**d.** Sketch the function.

**Solution:**

**a.**
$$= - \quad {}^2 + 2 \quad + 3$$
$$= -( \quad {}^2 - 2 \quad ) + 3$$
$$= -(( \quad - 1)^2 - 1) + 3$$
$$= -( \quad - 1)^2 + 1 + 3$$
$$= -( \quad - 1)^2 + 4$$

**b.** This is the function $= - \quad {}^2$ moved to right so that its axis of symmetry is

$= 1$ and shifted up by 4, i.e., its orthogonal axis is $= 4$.

**c.** The function is $= -(\ -1)^2 + 4$. This will cross the -axis when $= 0$

i.e. when $-(\ -1)^2 + 4 = 0$

$$4 = (\ -1)^2$$

taking square roots yields $-1 = \pm 2$ implies $x = 1 \pm 2$, i.e.,

$= -1$ or $3$. Putting $= 0$ into the original form of the function at the

$= -\ ^2 + 2\ + 3$. This gives $= 3$, i.e., it crosses the -axis at $= 3$.

**d.** The function is shown in figure1.34.

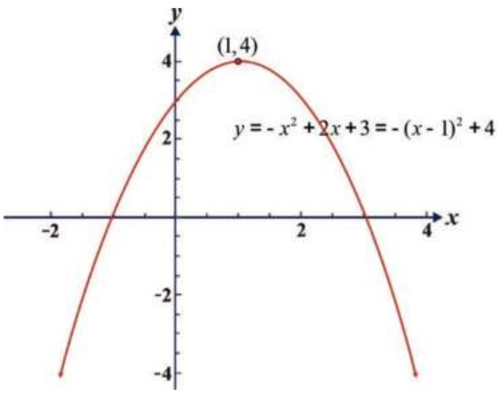



Figure 1.34 the graph of $= -\ ^2 + 2\ + 3$

## Exercise 1.23

1. A quadratic function is given as $= -\ ^2 - 2\ + 1$.

   a. completes the square on this function and find the vertex.

   b. use this to determine the axis of symmetry and the orthogonal axis of the curve.

   c. finds the points on the x and y axes where the curve crosses them.

   d. Sketch the function.

2. A quadratic function is given as $= -\ ^2 - 2\ - 1$.

   a. completes the square on this function and find the vertex.

   b. Sketch the function.

3. Sketch each of the following quadratic functions.

   a. $= 6 -\ ^2$      b. $= 4\ -\ ^2$

4. Find the and - intercepts, axis of symmetry and orthogonal axes of the following functions.

   a. $= -2\ ^{2} - 8$ b. $= -\ ^{2} + 6\ + 7$

**Note**

1. The graph of $f(\ ) = (\ + k)^2 + c$ opens upward.
2. The graph of $f(\ ) = -(\ + k)^2 + c$ opens downward.
3. The vertex of the graph of $f(\ ) = (\ + k)^2 + c$ is $(-k, c)$ and the vertex of the graph of $f(\ ) = (\ - k)^2 - c$ is $(k, -c)$. Similarly, the vertex of the graph of $f(\ ) = (\ + k)^2 - c$ is $(-k, -c)$ and the vertex of the graph of $f(\ ) = (\ - k)^2 + c$ is $(k, c)$.

# 1.3 Applications of Relations and Functions

## Applications involving relations

### Example 1

The data in table1.13 depicts the length of a woman's femur and her corresponding height. Based on these data, a forensics specialist can find a linear relationship between heights inch and femur $cm$: $= 2{,}47\ + 54,\ 40 \leq\ \leq 55$.

Table 1.13

| Length of femur (cm) | Height (cm) |
|---|---|
| 45 | 164 |
| 48 | 173 |
| 42 | 158 |
| 46 | 167 |
| 50 | 178 |

From this type of relation, the height of a woman can be inferred based on skeletal remains.

**a.** Find the height of the woman whose femur is 44 $cm$.

**b.** Find the height of the woman whose femur is 51 $cm$.

**Solution:**

**a.** $= 2.47 \quad + 54$

$= 2.47(44) + 54$

$= 162.68$

The woman is approximately 163 cm tall.

**b.** $= 2.47 \quad + 54$

$= 2.47(51) + 54$

$= 179.97$

The woman is approximately 180 cm tall.

## Example 2

If the equation $^2 + 18 \quad + 81$ represents the area of the square, what is the perimeter of the square if $\quad = 10$?

**Solution:**

$^2 + 18 \quad + 81$ factors into $( \quad + 9)( \quad + 9)$

Since this represents the area of a square where length = width, then

$( \quad + 9) =$ length and $( \quad + 9) =$width

Perimeter of a square = 2(length + width) = $2( \quad + 9 + \quad + 9)$

$= 2(2 \quad + 18) = 4 \quad + 36$

for $\quad = 10$, Perimeter $4(10) + 36 = 40 + 36 = 76$

## Example 3

The width of a square is 1 less than twice its width. What is its length?

**Solution:**

When 1 less than a number, the algebraic symbol is is $\quad - 1$

Let the width be

when  is 1 less than 2

$1 = 2 \ -$ and $= 1$. In a square, length = width. So, the length is 1.

## Exercise 1.24

1. There is 8m wire. With this wire, make a rectangle with length  m on one side.
   a. Express the width using .
   b. Express the area $S$ of the rectangle using .
   c. If $f(\ ) = S$, draw the graph $f(\ )$.
   d. Determine the domain and range of $f(\ )$.
2. There is an isosceles triangle with height 4cm.
   a. If the base is  cm and express the area S using .
   b. If $f(\ ) = S$, draw the graph of $f(\ )$.
   c. Determine the domain and range of $f(\ )$.

## Minimum and Maximum values of quadratic functions

Suppose you throw a stone upward. The stone turns down after it reaches its maximum height. Similarly, a parabola turns after it reaches a maximum or a minimum $y$ value.

### Activity 1.10

Let $f(\ )$ be a quadratic function. Discuss how to determine the maximum or minimum value of $f(\ )$.

### Example 1

The minimum value of a quadratic function expressed as

$f(\ ) = (\ \ + k)^2 + c$ is $c$.

Similarly, the maximum value of $f(\ ) = -(\ \ + k)^2 + c$ is $c$.

## Example 2

Find the maximum value of the function $f(\ ) = -\ ^2 + 6\ \ - 8$, and sketch its graph.

**Solution:**

$f(\ ) = -\ ^2 + 6\ \ - 9 + 9 - 8$

$= -(\ ^2 - 6\ \ + 9) + 1;$

$f(\ ) = -(\ \ - 3)^2 + 1.$

The graph of $f(\ ) = -(\ \ - 3)^2 + 1$ has vertex $(3, 1)$ and hence the maximum value of $f$ is 1.

In this case, the range of the function is $\{\ :\ \ \leq 1\} = (-\infty, 1]$

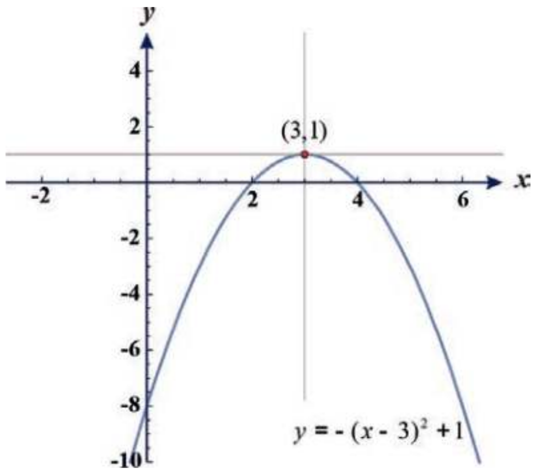



Figure 1.35 the graph of $f(\ ) = -(\ \ - 3)^2 + 1$

## Exercise 1.25

1. Find the vertex and the axis of symmetry of the following functions
   a. $f(\ ) = (\ \ - 4)^2 - 3$
   b. $f(\ ) = \ \ ^2 - 5\ \ + 8$
2. Determine the minimum or the maximum value of each of the following functions and draw the graphs:
   a. $f(\ ) = \ \ ^2 + 4\ \ + 1$
   b. $f(\ ) = 4\ ^2 + 2\ \ + 4$
   c. $f(\ ) = -\ \ ^2 - 4$
   d. $f(\ ) = -6 - \ \ ^2 - 4$
3. A metal wire 40 cm long is cut into two and each piece is bent to form a square. If the sum of their areas is 58 sq.cm, how long is each piece?

## Graphical method of solving quadratic equations

### Activity 1.11

For the quadratic function: $\ = \ {}^2$, complete the table, and plot the pair of coordinates on the coordinate plane to draw the graph.

Table 1.14

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | … | -3 | -2 | -1 | 0 | 1 | 2 | 3 | … |
| | … | 9 | 4 | 1 | 0 | | | | |

In general, the graph of quadratic function has u-shaped curve and one extreme point, called vertex. In the Activity, the vertex of the function is $(0, 0)$. In order to find the solutions of a quadratic equation using a graph:

1. Rearrange the equation so that one side is equal to zero (if necessary).
2. Draw the graph of the quadratic function.
3. Read off the  -coordinate(s) of the point(s) where the curve crosses the  -axis.

   The quadratic function is transformed using completing the square method as follows:

$$= ax^2 + bx + c = a\left( \ \ + \frac{b}{2a}\right)^2 + c - a\cdot\frac{b^2}{4a^2} = a\left( \ \ + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$= a\left( \ \ + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Hence, the vertex of the function is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

(That is, translate $\ = ax^2$ by $-\frac{b}{2a}$ in  -axis direction, and by $\frac{4ac-b^2}{4a}$ in  -axis direction.

It uses the vertex formula $\left(-\frac{b}{2a}, \ \frac{4ac-b^2}{4a}\right)$ to get the vertex which gives an idea of what values to choose to plot the points. If the graph of the quadratic function crosses the  -axis at two points then we have two solutions. The  -intercept(s) of a graph is/are the solution(s) of the equation. If the graph touches the  -axis at one point then we have one solution. If the graph does not intersect with the -axis then the equation has no real solution.

### Example 1

Find the solutions of the equation $^{2}-4=0$ graphically.

**Solution:**

Draw the graph of the quadratic function, $= {}^{2}-4$

Read off the $x$-coordinate(s) of the point(s) where the curve crosses the $x$-axis.

The roots are $=-2$ and $=2$

These are the solutions of $^{2}-4=0$

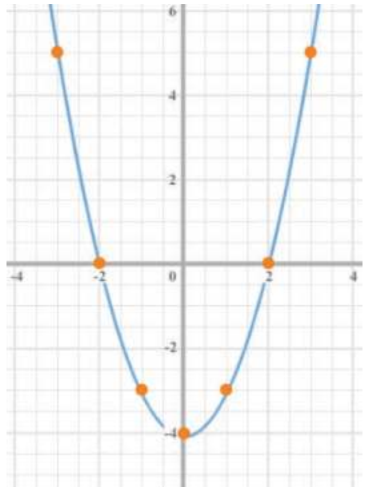


Figure 1.36 the graph of $= {}^{2}-4$

### Example 2

Solve $^{2}-2 \quad -3=0$ graphically.

**Solution:**

The solution for the equation $^{2}-2 \quad -3=0$ can be obtained by looking at the points where the graph $= {}^{2}-2 \quad -3$ cuts at the -axis (i.e., $=0$).

$= {}^{2}-2 \quad -3=( \quad -1)^{2}-4,$ hence the vertex is $(1,-4)$

In addition to the vertex, plot some points, such as $y$-intercept $(0,-3)$ and other points $(-1,0),(2,-3)$, $(3,0)$ to draw graph.

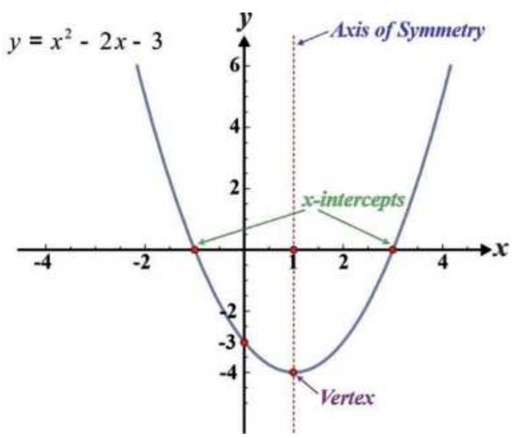



Figure 1.37 the graph of $= {}^{2}-2 \quad -3$.

The graph $\ \ = {}^2 - 2\ \ - 3$ cuts the  -axis at $\ \ = -1$ and $\ \ = 3$. So, the solution for the equation is $\ \ = -1$ and $\ \ = 3$.

## Example 3

Solve $-\ \ {}^2 + 6\ \ - 9 = 0$ graphically.

**Solution:**

The solution for the equation $-\ \ {}^2 + 6\ \ - 9 = 0$ can be obtained by looking at the points where the graph $\ \ = \ \ {}^2 + 6\ \ - 9$ cuts at the  -axis (i.e., $\ \ = 0$).

$\ \ = -\ \ {}^2 + 6\ \ - 9 = -(\ \ - 3)^2$, hence the vertex is $(3, 0)$

Plot other points, y-intercept $(0, -9)$, and $(1, -4)$, $(2, -1)$, (4, -1) to draw graph.

The graph $\ \ = -\ \ {}^2 + 6\ \ - 9 = 0$ cuts the $\ \ - axis$ only at $\ \ = 3$. So, the solution for the equation is $\ \ = 3$.

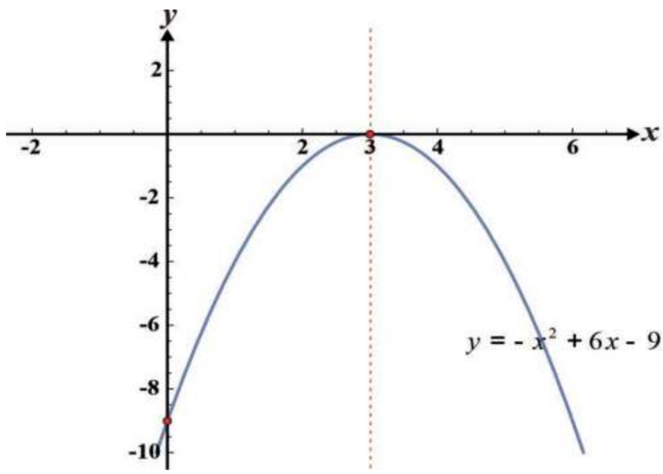



Figure 1.38 the graph of the function $\ \ = -\ \ {}^2 + 6\ \ - 9$

## Example 4

Solve $\ \ {}^2 + 4\ \ + 8 = 0$ graphically.

**Solution:**

The vertex for $\ \ = \ \ {}^2 + 4\ \ + 8$ is given by using completing the square method as follows:

$$= \ ^2 + 4 \quad + 8 = ( \quad + 2)^2 + 4$$

Then the vertex is (-2, 4). Furthermore, coefficient of $\ ^2$ is positive. Therefore, there is no -intercept and hence, no real solution to the equation $\ ^2 + 4 \quad + 8 = 0$

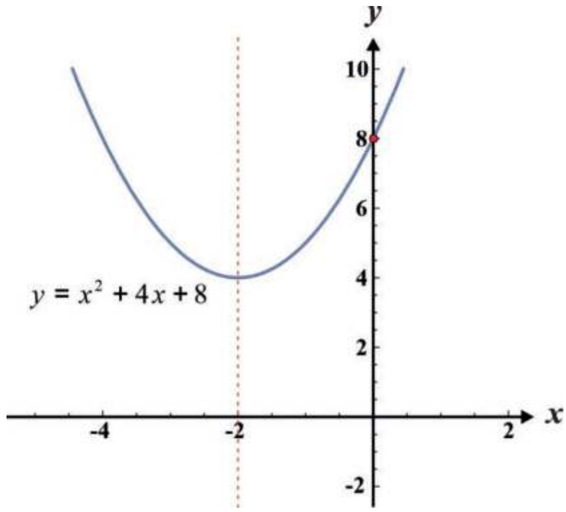



Figure 1.39 the graph of the function $\quad = \ ^2 + 4 \quad + 8$

**Note**

In the graphs of quadratic functions, the sign on the coefficient $a$ affects whether the graph opens up or down. If $a > 0$, then the graph opens up and if $a < 0$, then the graph opens down.

## Exercise 1.26

1. Use graphical method to solve the following.

   a. $\ ^2 - 1 = 0$ b. $\ ^2 + 2 \quad + 1 = 0$

   c. $\ ^2 + 3 \quad - 4 = 0$ d. $\ ^2 - 4 \quad + 6 = 0$

2. The quadratic function f intersects the -axis at the points $(1, 0)$ and $(-4, 0)$. What is the solution set of the equation $f( \ ) = 0$?

3. At what values of does the graph of the equation $\quad = ( \quad + 2)( \quad - 6)$ cross the x-axis?

## Graphical method of solving quadratic inequalities (1)

### Activity 1.12

Given the following inequalities:

a. $> \ ^{2}$ b. $< \ ^{2}$ c. $> \ ^{2} + 4$

d. $\geq 1 - \ ^{2}$ e. $\leq \ ^{2} + 2$ f. $\geq \ ^{2} - 4$

1. Find the -intercept and -intercept of their respective equation.
2. Which region (inside or outside the parabola satisfy the inequality)?

**In solving a quadratic inequality graphically, the following steps shall be followed.**

1. Write the quadratic inequality in standard form $\ ^{2} + bx + c \geq 0$ $or$ $a \ ^{2} + bx + c \leq 0$.
2. Graph the function $f(\ ) = a \ ^{2} + bx + c$ using properties or transformations.
3. Determine the solution from the graph.

The graph of a quadratic function $f(\ ) = a \ ^{2} + bx + c$ is a parabola. When we ask $a \ ^{2} + bx + c < 0$, $f(\ ) < 0$ (Figure 1.40). We want to know when the parabola is below the $x$ -axis. When we ask $a \ ^{2} + bx + c > 0$, we are asking when $f(\ ) > 0$ (Figure 1.41). We want to know when the parabola is above the $x$ -axis.

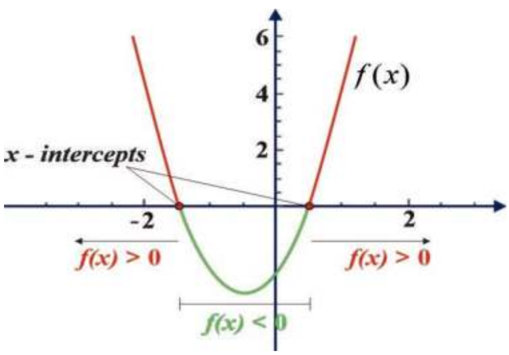



Figure 1.40 the graph of the function $= ax^2 + bx + c$ for $a > 0$

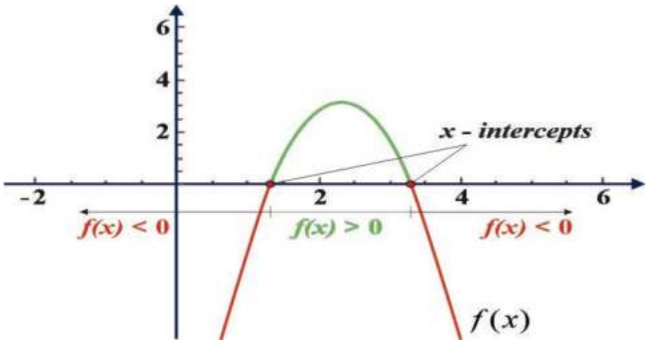



Figure 1.41 the graph of the function $\quad = ax^2 + bx + c$ for $a < 0$

## Example 1

Solve $\ ^2 - 6 \quad + 8 < 0$

### Solution:

Step 1: Write the quadratic inequality in standard form. The inequality is in standard form.

Step 2: Graph the function $f(\ ) = a \ ^2 + bx + c$ using properties or transformations.

We will graph using the properties.

Look at $a$ in the equation $f(\ ) = \ ^2 - 6 \quad + 8$. $a = 1, b = -6,\ and\ c = 8$.

Since a is positive, the parabola opens upward.

$f(\ ) = \ ^2 - 6 \quad + 8 = (\quad - 3)^2 - 1$. The vertex is $(3, -1)$. Using table to find other points.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(\ ) = \ ^2 - 6 \quad + 8$ | 3 | 0 | $-1$ | 0 | 3 |

The  -intercepts are $(2,0)$ and $(4, 0)$ . We graph the  -intercepts, and the *y*-intercept. We connect these points to sketch the parabola (Figure 1.42).

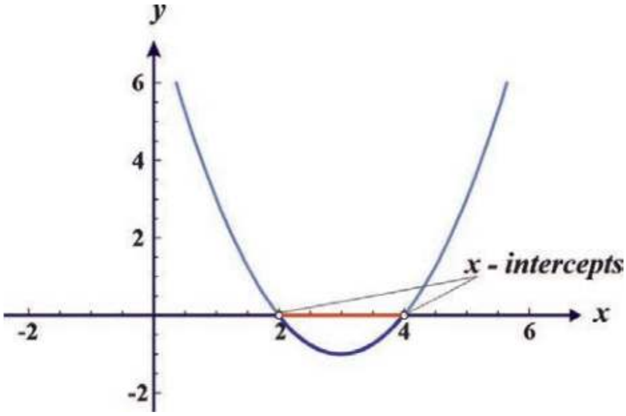



Figure 1.42 the graph of the function $= {}^2 - 6 \quad + 8$

**Step 3:** Determine the solution from the graph.

The inequality asks for the values of which make the function less than 0. Which values of make the parabola below the -axis? We do not include the values 2, 4 as the inequality is less than only. The solution, in interval notation is $(2, 4)$ or $\{ \ : 2 < \quad < 4\}$ for the quadratic inequality $^2 - 6 \quad + 8 < 0$.

## Exercise 1.27

Solve the following quadratic inequalities graphically. Use the previous methods to check the answers.

a. $^2 + 6 \quad + 5 < 0$

b. $^2 + 6 \quad - 7 \geq 0$

c. $^2 - 10 \quad + 16 > 0$

d. $^2 - 3 \quad + 4 > 0$

e. $^2 - 2 \quad - 15 \leq 0$

## Graphical method of solving quadratic inequalities (2)

### Example 2

Solve $- \ ^2 - 8 \quad - 12 \leq 0$ graphically. Write the solution in interval notation.

**Solution:**

Consider the equation $f(\ ) = - \ ^2 - 8 \quad - 12$. $a = -1, b = -8,$ *and* $c = -12$.

Since $a$ is negative, the parabola opens downward.

We find $f(0) = -12$. Hence $(0, -12)$ is the -intercept.

$f(\ ) = -\ ^2 - 8\ \ - 12 = -(\ ^2 + 8\ \ + 12)$

$= -(\ \ + 4)^2 + 4.$

The vertex is $(-4,\ 4)$. Find other points using the table.

| | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ |
|---|---|---|---|---|---|
| $f(\ ) = -\ ^2 - 8\ \ - 12$ | 0 | 3 | 4 | 3 | 0 |

So, the *x*-intercepts are $(-6, 0)$ and $(-2, 0)$. We graph the vertex, intercepts. We connect these points to sketch the parabola (Figure 1.43).

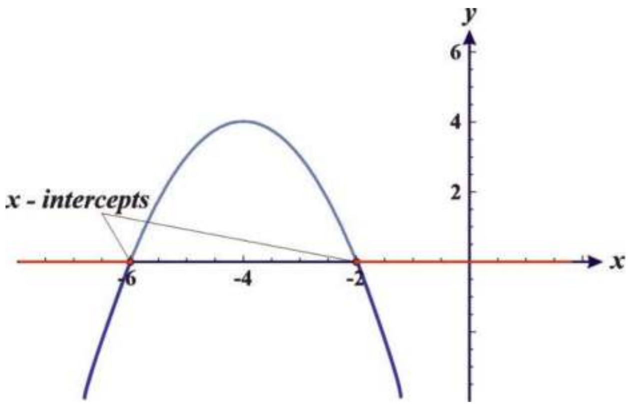



Figure 1.43 the graph of the function $\quad = -\ ^2 - 8\ \ - 12 \leq 0.$

The inequality asks for the values of   which make the function less than or equal to 0. We include the values $-6$,and $-2$ as  the inequality is less than or equal to 0. The solution, in interval notation is $(-\infty, -6] \cup [-2, \infty)$ or $\{\ :\ \ \leq -6\ or\ \ \geq -2\}$ for the quadratic inequality

$-\ ^2 - 8\ \ - 12 \leq 0.$

## Exercise 1.28

Solve the following quadratic inequalities graphically. Use the previous methods to check the answers.

a. $-\ ^2 +\ \ + 12 \geq 0$  b. $-\ ^2 - 2\ \ + 17 < 0$

c. $-\ ^2 - 6\ \ - 5 < 0$  d. $-\ ^2 + 2\ \ + 1 \geq 0$

e. $-\ ^2 + 8\ \ - 14 < 0$

# Summary

1. A Cartesian coordinate system in two dimensions (also called a rectangula**r** **coordinate** system) is defined by an ordered pair of perpendicular lines (axes), a single unit of length for both axes, and an orientation for each axis.
2. In relation, two things are related to each other by a relating phrase.
3. A relation is the set of ordered pairs.
4. The set of first components in the ordered pairs is called the **domain** of the relation. The set of second components in the ordered pairs is called the **range** of the relation.
5. A function is a special type of relation in which no two distinct ordered pairs have the same first element.
6. A function from $A\ to\ B$ can sometimes be denoted as $f: A \rightarrow B$, where the domain of the function $f$ is $A$ and the range of the function $f$ is included in $B$, in which we say $B$ contains the image of the elements of $A$ under the function $f$.
7. A numerical relation $R$ is a function if and only if no **vertical line** in the plane intersects the graph of $R$ in more than one point.
8. If each element in the domain of a function has a distinct image in the co-domain, then function is said to be one -to- one function.
9. A function is called an onto function if each element in the co-domain has at least one pre – image in the domain.
10. A function $f: A \rightarrow B$ is said to be a one- to-one correspondence if $f$ is both one to one and onto.
11. Let $f$ and $g$ be two functions with overlapping domains. We define the sum, difference, product, and quotient as:

$$(f+g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x)$$

$$(f-g)(x) = f(x) - g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

**12.** If $a$ and $b$ are fixed real numbers, $a \neq 0$,then $f(x) = ax + b$ for every real number $x$ is called a linear function.

**13.** In $f(x) = ax + b$ for $a \neq 0, x\epsilon/R$, $a$ represent the slope, $(0, b)$ represents the $y$- intercept and $(-\frac{b}{a}, 0)$ represent the $x$- intercept.

**14.** A function defined by $f(x) = ax^2 + bx + c$ where $a, b, c$ are real numbers and $a \neq 0$ is called a quadratic function. The point $a$ is the leading coefficient of $f$.

**15.** We can sketch the graph of a linear function by using either a table of values, or the $x$ and $y -$ intercepts.

**16.** We can sketch the graph of a quadratic function by using either a table of values, or shifting rule.

**17.** The graph of $f(x) = ax^2 + bx + c$ opens upward if $a > 0$ and downward if $a < 0$.

**18.** The vertex is the point on the coordinate system at which a graph of a quadratic function turns either upward or downward.

**19.** The axis of symmetry is a vertical line that passes through the vertex of the parabola.

**20.** The domain and range of linear functions are the set of real numbers.

**21.** The domain of a quadratic function $f(x) = ax^2 + c$ is the set of real numbers, whereas the range is;

$\{y: y \geq c\}$ if the leading coefficient is positive and c is the value of $y$ at the vertex.

$\{y: y \leq c\}$ if the leading coefficient is negative and $c$ is the value of $y$ at the vertex.

*For* $f(x) = ax^2 + bx + c$. In this case, $f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2-4ac)}{4a}$. The coordinate of the vertex is $\left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a}\right)$. If $a > 0$, the range is

$\{y: y \geq \frac{b^2-4ac}{4a}\}$. If $a < 0$, the range is$\{y: y \leq \frac{b^2-4ac}{4a}\}$.

## Review Exercise

**1.** Which of the following sets of ordered pairs are functions?

**a.** $(1, 2), (2, 3), (3, 4), (4, 5)$

**b.** $(4, 3), (2, 2), (-3, 4), (-3, -3)$

**c.** $(1, 2), (2, 3), (1, 3), (4, 5)$

**d.** $(1, -1), (1, -6), (4, 2), (2, -3)$

**2.** Let $R = \{(x,\ y): y\ is\ shorter\ than\ x\}$.

**a.** Does$(x, x)$ belong to the relation? Why?

**b.** Is true that if $(x, y)$ belongs to $R$, then $(y, x)$ belongs to $R$?

**c.** If $(x, y)$ and $(y, z)$ belong to $R$, then is it true that $(x, z)$ belongs to $R$?

**3.** Find the domain and range of each of the following relations:

**a.** R is the relation which contains the set of order pairs $(x, y): y = -3x$.

**b.** R is the relation which contains the set of order pairs $(x, y): x$ contains $-2, -1, 0, 1, 3, 5$ and $y = 2 - x$

**c.** R is the relation which contains the set of order pairs $(x, y): y = \sqrt{1 - x^2}$.

**d.** The set of ordered pairs $(x, y)$, where $y$ is a sister of $x$.

**e.** The set of ordered pairs $(x, y)$, where a pupil in $y's$ class is $x$.

**4.** Sketch the graph of each of the following relations and find the domain and range.

**a.** R is the relation which contains the set of order pairs $(x, y): y = x^3$.

**b.** R is the relation which contains the set of order pairs $(x, y): y \leq -x - 1$.

**c.** R is the relation which contains the set of order pairs $(x, y): y > 3x - 1$.

**d.** R is the relation which contains the set of order pairs $(x, y): y \geq 2x - 3$.

**e.** R is the relation which contains the set of order pairs $(x, y): y \geq 2x$ and $y < x + 1$.

**f.** R is the relation which contains the set of order pairs $(x, y): y < -x + 2$ and $y > x - 3$.

5. For the following graph (Figure 1.44), specify the relation and write down the domain and range:

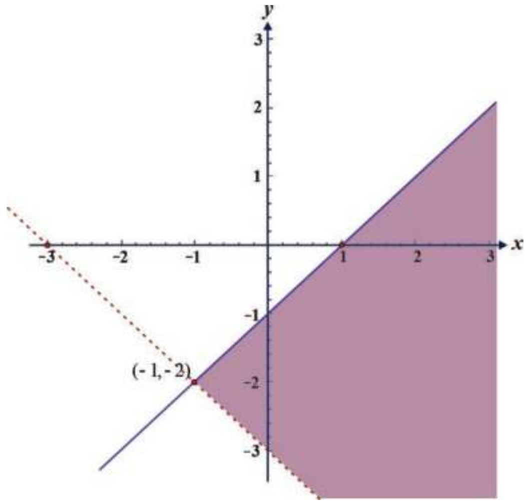



Figure 1.44

6. Let $f(x) = 2x^2 - 5x - 3$ and

$g(x) = -2x^2 + x + 7$

a. Find:

i) $f + g$ ii) $f - g$

iii) $(f + g)(-1)$ iv) $(2f - g)(3)$

b. Determine the domain of $f - g$.

7. Let $f(x) = 2x^2 - 1$ and

$g(x) = x - 3$, then

a. Evaluate: i) $fg$ ii) $\frac{f}{g}$ iii) $(fg)(2)$ iv) $\left(\frac{f}{g}\right)(5)$

b. Find the domain of $\frac{f}{g}$.

8. If $f(x) = \frac{2x-3}{x-1}$ and $g(x) = \frac{x+8}{x}$, then

a. find: i) $fg$ ii) $\frac{f}{g}$ iii) domain of $fg$ *and* $\frac{f}{g}$

b. evaluate: i) $(fg)(-3)$ ii) $\left(\frac{f}{g}\right)(3)$ iii) $\left(3f - \frac{f}{g}\right)(-1)$

9. Determine which of the following pairs of formulas define the same function.

a. $y = x^2$ ; $y = x^2$ , $x \geq o$.

b. $y = (x + 1)^2(x - 2)$ ; $y = x^3 - 3x - 2$ .

c. $y = \frac{1}{x}$ ; $y = \frac{x+1}{x^2+x}$.

d. $y = 2^x$; $y = x$ .

e. $y = x + 1$; $y = \frac{x^2-1}{x-1}$.

f. $y = x$; $y = \sqrt{x^2}$

g. $y = x$, $y = \sqrt[3]{x^3}$.

h. $y = \frac{2x+1}{x}$ ; $y = \frac{2x^2+7x+3}{x^2+3x}$, $x > 0$.

**10.** Construct tables of values and sketch each of the following:

**a.** $f(x) = 2x - 5$ **b.** $2x + y - 3 = -5$

**c.** $f(x) = 7 - 6x - x^2$ **d.** $f(x) = 1 - 4x$

**11.** The curve $y = x^2$ is shifted so that its axis of symmetry is the line $x = 1$ and its orthogonal axis is $y = -4$.

**a.** Write down the equation of the new curve.

**b.** Find the coordinates of the points where this new curve cuts the $x$ and $y$ axes.

**c.** Sketch the curve

**12.** The curve $y = x^2$ is shifted so that its axis of symmetry is the line $x = 1$ and its orthogonal axis is $y = 3$.

**a.** Write down the equation of the new curve.

**b.** Find the coordinates of the points where this new curve cuts the $x$ and $y$ axes.

**c.** Sketch the curve

**13.** The curve $y = -x^2$ is shifted so that its axis of symmetry is the line $x = 1$ and its orthogonal axis is $y = 3$.

**a.** Write down the equation of the new curve.

**b.** Find the coordinates of the points where this new curve cuts the $x$ and $y$ axes.

**c.** Sketch the curve

**14.** By using shifting rule, sketch the graph of each of the following:

**a.** $y = x^2 - 4x + 2$

**b.** $y = -x^2 - 6x - 8$

**15.** A mobile phone technician uses the linear function $C(t) = 2t + 100$ to determine the cost of repair where the time in the hours is $t$ and $C(t)$ is the cost in Birr. How much will you pay if it takes him 3 hours to repair your mobile?

# UNIT 2

# POLYNOMIAL FUNCTIONS

## Unit Outcomes

By the end of this unit, you will be able to:

- Define polynomial functions.
- Perform the four fundamental operations on polynomials.
- Apply theorems on polynomial functions to solve related problems.
- Determine the rational and irrational zeros of polynomials.
- Sketch the graphs of polynomial functions.

## Unit Contents


- ✓ Linear function
- ✓ Domain
- ✓ Location Theorem
- ✓ Turning point
- ✓ Zero(s) of polynomial
- ✓ $x$-intercept
- ✓ Remainder theorem
- ✓ Polynomial Division Theorem
- ✓ $y$-intercept
- ✓ Factor theorem
- ✓ Polynomial Function
- ✓ Multiplicity
- ✓ Leading Coefficient
- ✓ Rational root test
- ✓ Degree
- ✓ Constant function
- ✓ Constant term

## Introduction

In unit one of this textbook, you saw functions of the form $y = b, y = ax + b$ and $ax^2 + bx + c$. You also attempted to sketch their graphs. These functions are parts of a large class of functions called polynomial functions. Polynomial functions are functions that involve only one variable $x$, consisting of the sum of several terms; each term is a product of two factors; the first being a real number coefficient and the second being $x$ raised to some non-negative integer power. In this unit, you will be looking at the different components of polynomial functions like degree, leading coefficient, zeros of a polynomial function, theorems on polynomial functions and properties of graphs of polynomial function. You will see how the leading coefficient and the degree of a polynomial function determine the end property of the graph of the function.

## 2.1 Definition of Polynomial Function

You are familiar with functions like constant functions, linear functions and quadratic functions in unit one.

## Activity 2.1

Classify the following functions as constant function, linear function, quadratic function or none of these:

a. $f(x) = 2x + 4$

b. $g(x) = \frac{2}{3} - x$

c. $f(x) = |x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$

d. $h(x) = -x^2 + 5x + 9$

e. $f(x) = x^4 + 2x - 5$

f. $k(x) = 5$

g. $h(x) = (x - 2)(x + 2)$

h. $f(x) = 6 + \frac{9}{5}x + 4x^2$

i $f(x) = -\sqrt{5}x + \sqrt{2}$

j. $l(x) = 3\sqrt{x} + \sqrt{3}$

A function $f$ is a constant function if it can be written in the form $f(x) = b$, where $b$ is a real number. The domain of $f$ is the set of all real numbers and the range is the set containing only the number $b$.

A function $f$ is a linear function if it can be written in the form $f(x) = ax + b$, where $a$ and $b$ are real numbers and $a \neq 0$. The domain of $f$ is the set of all real numbers and the range is also the set of all real numbers.

A function $f$ is a quadratic function if it can be written in the form $f(x) = ax^2 + bx + c$ where $a$, $b$ and $c$ are real numbers and $a \neq 0$. The domain of $f$ is the set of all real numbers and the range is not the set of all real numbers and it depends on the values of $a, b$ and $c$.

## Definition of Polynomial Function

### Definition 2.1

Let $n$ be non-negative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called a polynomial function in one variable $x$ of degree $n$.

In the above definition of polynomial function

i. $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are called the **coefficients** of the polynomial function (or simply the polynomial).

ii. The number $a_n$ is called the **leading coefficient** of the polynomial and $a_n x^n$ is the **leading term**.

iii. The number $n$ (the exponent of the highest power of $x$) is the **degree** of the polynomial.

iv. The number $a_0$ is called the constant term of the polynomial.

**Domain of Polynomial function**: The domain of a polynomial function is the set of all real numbers.

**Note**

- The constant function $f(x) = b, b \neq 0$ is polynomial function with degree zero.
- The constant function $f(x) = 0$ is called the zero polynomial with no degree assigned to it.

### Example 1

Find the degree, leading coefficient and constant term of the following polynomial functions.

**a.** $f(x) = -2x + 5$

**b.** $f(x) = 3x^2 + x - 3$

**c.** $f(x) = 3x^3 - 9x^2 + 5x + \frac{3}{2}$

**d.** $h(x) = -5x^4 + 8x^3 + 2x^2 - 3x + 7$

**e.** $g(x) = -2 + 3x^3 + \frac{2}{5}x^2 - x^4 + 4x^5 + 5x$

**f.** $g(x) = 2(x^4 + \frac{1}{2}x^2 - 2) + x + x^4 + 1$

**Solution:**

**a.** It is a polynomial function with degree 1, leading coefficient -2 and constant term 5.

**b.** It is a polynomial function with degree 2, leading coefficient 3 and constant term $-3$.

**c.** It is a polynomial function with degree 3, leading coefficient 3 and constant term $\frac{3}{2}$.

**d.** It is a polynomial function with degree 4, leading coefficient -5 and constant term 7.

**e.** You can rearrange the polynomial function $g$ as

$g(x) = 4x^5 - x^4 + 3x^3 + \frac{2}{5}x^2 + 5x - 2$ and it is a polynomial function with degree 5, leading coefficient 4 and constant term $-2$.

**f.** $g(x) = 2(x^4 + \frac{1}{2}x^2 - 2) + x + x^4 + 1 = 2x^4 + x^2 - 4 + x + x^4 + 1$

$$= 3x^4 + x^2 + x - 3.$$

Therefore, the degree is 4, the leading coefficient is 3 and the constant term is $-3$.

## Exercise 2.1

Find the degree, leading coefficient and constant term of the following polynomial functions.

a. $f(x) = -x^2 + 3x + 9$

b. $f(x) = 12x^3 - 9x^2 + 3x + 4$

c. $h(x) = -7x^5 + 5x^3 - 10$

d. $g(x) = -4x^2 - x^3 - 5 + 2x + 3x^4$

e. $h(x) = -3x^4 + x^2 + 3\,(2x^2 + 4x^4 + 5x^3 + \frac{2}{3})$

**Note**

There are two ways to check whether a given function is not polynomial function.

- Its domain is not all real numbers.
- It does not take the form or unable to transform it to the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $n$ is non-negative integer ( 0 or positive integers) and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers with $a_n \neq 0$.

### Example 2

Determine if the following are polynomial function or not.

**1.** $g(x) = 4x^{-2} + 3x^{-1} - 7$ **2.** $h(x) = 2^x$

**Solution:**

**1.** $g(x) = 4x^{-2} + 3x^{-1} - 7$ is not a polynomial function because $-2$ and $-1$ are not positive integers.

**2.** $h(x) = 2^x$ is not a polynomial function because it can't take the form of the general polynomial function.

### Example 3

Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term.

**a.** $f(x) = 2x^2 - 9x^3 + \frac{2}{5}x$ **b.** $g(x) = (2 - x)(2 + x)$`

**c.** $h(x) = 5x^{-4} + 6x^2 + 3$ **d.** $k(y) = \sqrt{y}$

**e.** $f(x) = 2x^{\frac{1}{3}} + 5x^{\frac{1}{2}} - x + 5$ **f.** $f(x) = \xi\,\overline{2}x^3 + 5x^2 - \sqrt{\frac{2}{7}}$

**Solution:**

**a.** You can rearrange it as $f(x) = -9x^3 + 2x^2 + \frac{2}{5}x$ and it is a polynomial function with degree 3, leading coefficient $-9$ and constant term 0.

**b.** $g(x) = (2 - x)(2 + x) = 4 - x^2$ , it is a polynomial function with degree 2, leading coefficient $-1$ and constant term 4.

**c.** $h(x) = 5x^{-4} + 6x^2 + 3$ is not a polynomial function because in the term $5x^{-4}$, $n = -4$ is not a positive integer.

**d.** $k(y) = \sqrt{y}$ is not a polynomial function because it cannot be written in the general form of polynomial function, because the variable $y$ is inside a radical sign.

**e.** $f(x) = 2x^{\frac{1}{3}} + 5x^{\frac{1}{2}} - x + 5$ is not a polynomial function because $\frac{1}{3}$ and $\frac{1}{2}$ are not integers.

**f.** It is a polynomial function with degree 3, leading coefficient $\sqrt{2}$ and constant term $-\sqrt{\frac{2}{7}}$.

## Exercise 2.2

Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient and constant term.

a. $f(x) = 3x^5 + 2x^3 + x - 5$

b. $k(x) = 4 - x + \frac{3}{8}x^4 - 8x^2$

c. $g(x) = 4x^{-3} - 7x^{-2} + x^{-1} - 8$

d. $h(t) = \sqrt{2}t + \sqrt{3}$

e. $h(y) = \sqrt{6}y^2 + \sqrt{2}y + 5$

f. $f(x) = \dfrac{1}{x^2} + \dfrac{1}{x} + 1$

g. $k(x) = 2x^5 - 5x^2 + 2 - \dfrac{2 - x^2 + 4x^3 + x^5}{3}$

h. $f(x) = (x+2)(x^2 + x - 1)$

i $f(x) = \sqrt{(x^2+5)^4}$

j. $f(x) =$ ä$2x^2 + 7$à

## Definition 2.2

A polynomial expression is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ where $n$ is non negative integer and $a_n \neq 0$. Each individual expression $a_kx^k$ making up the polynomial is called a term.

## Example 4

Consider the expression $\frac{8-3x^3+4x^4}{4} - 3x^4 + 5x^2$.

**a.** Is it a polynomial expression?

**b.** Find the degree, leading coefficient and the constant term.

**c.** What is the coefficient of $x^3$ ?

### Solution:

First, write $\frac{8-3x^3+4x^4}{4} - 3x^4 + 5x^2$ in the general form of a polynomial expression,

$$\frac{8-3x^3+4x^4}{4} - 3x^4 + 5x^2 = \frac{8}{4} - \frac{3}{4}x^3 + \frac{4}{4}x^4 - 3x^4 + 5x^2$$

$$= 2 - \frac{3}{4}x^3 + (x^4 - 3x^4) + 5x^2 \;\; = \; 2 - \frac{3}{4}x^3 - 2x^4 + 5x^2$$

$$= -2x^4 - \frac{3}{4}x^3 + 5x^2 + 2.$$

**a.** Yes, it is a polynomial expression.

**b.** The degree is 4, the leading coefficient is $-2$ and the constant term is 2.

**c.** $-\frac{3}{4}$

## Exercise 2.3

Consider the expression $\frac{6x^3-2x^2+9}{3} + 3x^3 - 2x$

a. Is it a polynomial expression?

b. Find the degree, leading coefficient and constant term.

c. What is the coefficient of $x^2$?

# 2.2 Operations on Polynomial Functions

Recall in algebra that we can combine two real numbers using the operations addition, subtraction, multiplication and division to find another real number. You know the restriction we have to make when we use the operation division. Here, we will combine two or more polynomial functions using the operations addition, subtraction, multiplication and division and discuss the results obtained by the combination. To combine polynomial functions the knowledge of commutative, associative and distributive laws and like and unlike terms is very important.

## Activity 2.2

1. Which of the following pairs contain like terms?

   a. $2x$ and $5x$     b. $5a^2$ and $6a^2$

   c. $x^2$ and $2x^3$     d. $3x^2$ and $2y^2$

   e. $3$ and $y$     f. $x^5$ and $6x^5$

2. For any three real numbers $a, b$ and $c$, determine whether each of the following statements is true or false. Give reason for your answer.

   a. $a + b = b + a$     b. $a - b = b - a$

   c. $(a + b) + c = a + (b + c)$     d. $(ab)c = a(bc)$

   e. $a(b + c) = ab + ac$     f. $a - (b + c) = a - b + c$

   g. $a - (b - c) = a - b + c$     h. $(a - b) + c = a + (-b + c)$

## Addition of Polynomial Functions

### Definition 2.3

The sum of two polynomial functions $f$ and $g$ is written as $f + g$ and is defined as: $(f + g)(x) = f(x) + g(x)$ for all real numbers $x$.

**Note**

The sum of two polynomial functions is found by adding the coefficients of like terms.

### Example 1

In each of the following, find the sum of $f(x)$ and $g(x)$.

a. $f(x) = 2x^4 + 5x^3 + x^2 + 9x + 4$ and $g(x) = -3x^3 + 5x^2 - 5x - 9$

b. $f(x) = \frac{1}{2}x^2 + \frac{4}{3}x - 5$ and $g(x) = \frac{3}{2}x^2 - \frac{2}{3}x + 3$

**Solution:**

**a.** $f(x) + g(x)$

$= (2x^4 + 5x^3 + x^2 + 9x + 4) + (-3x^3 + 5x^2 - 5x - 9)$

$= 2x^4 + (5x^3 - 3x^3) + (x^2 + 5x^2) + (9x - 5x) + (4 - 9)$

(Grouping like terms)

$= 2x^4 + 2x^3 + 6x^2 + 4x - 5$ (Adding like terms)

**b.** $f(x) + g(x) = (\frac{1}{2}x^2 + \frac{4}{3}x - 5) + (\frac{3}{2}x^2 - \frac{2}{3}x + 3)$

$= (\frac{1}{2}x^2 + \frac{3}{2}x^2) + (\frac{4}{3}x - \frac{2}{3}x) + (-5 + 3)$

(Grouping like term)

$= \frac{4}{2}x^2 + \frac{2}{3}x - 2 = 2x^2 + \frac{2}{3}x - 2$ (Adding like terms)

## Observation:

1. If $f(x)$ and $g(x)$ have different degrees, the degree of $f(x) + g(x)$ is the same as the degree of $f(x)$ or the degree of $g(x)$ whichever has the highest degree.
2. If $f(x)$ and $g(x)$ have the same degree, the degree of the sum may be lower than or equal to the common degree.
3. The sum of two polynomial functions is a polynomial function.

## Exercise 2.4

Find the sum of the polynomial functions $f(x)$ and $g(x)$.

a. $f(x) = 2x^3 - 3x^2 - 2x + 5$ and $g(x) = 5x^4 + 6x^3 - 7x^2 + 4x + 3$.

b. $f(x) = -x^4 + 2x^3 - 3x^2 - 3x + 2$ and $g(x) = 5 + 7x - 2x^2 - x^3 + x^4$.

c. $f(x) = -2x^5 + 2x^4 - x^3 + 2x^2 + 5x - 1$ and $g(x) = 2 + 4x - 5x^5 - 3x^4$.

d. $f(x) = \sqrt{2}x^4 + 2x^3 - 5x^2 + x - \sqrt{3}$ and

$g(x) = \sqrt{3} - 3x - x^2 + 2x^3 - 2\sqrt{2}x^4$.

## Subtraction of Polynomial Functions

### Definition 2.4

The difference of two polynomial functions $f$ and $g$ is written as $f - g$, and is defined as $(f - g)(x) = f(x) - g(x)$ for all real numbers $x$.

### Example 2

In each of the following, find $f - g$:

**a.** $f(x) = -2x^3 + 5x^2 + 3x + 2$ and $g(x) = -2x^3 + 4x^2 + 8x - 7$.

**b.** $f(x) = 6x^5 + 5x^4 + 2x^3 - x^2 + 4x - 3$ and $g(x) = 5x^4 + x^3 + 4x^2 - 3x - 3$.

**Solution:**

**a.** $f(x) - g(x) = (-2x^3 + 5x^2 + 3x + 2) - (-2x^3 + 4x^2 + 8x - 7)$

$= -2x^3 + 5x^2 + 3x + 2 + 2x^3 - 4x^2 - 8x + 7$

(Removing brackets)

$= (-2x^3 + 2x^3) + (5x^2 - 4x^2) + (3x - 8x) + (2 + 7)$

(Grouping like terms)

$= 0x^3 + 1x^2 - 5x + 9$ (Adding like terms)

$= x^2 - 5x + 9.$

**b.** $f(x) - g(x) = (6x^5 + 5x^4 + 2x^3 - x^2 + 4x - 3\ ) - (5x^4 + x^3 + 4x^2 - 3x - 3)$

$= 6x^5 + 5x^4 + 2x^3 - x^2 + 4x - 3 - 5x^4 - x^3 - 4x^2 + 3x + 3$

(Removing Brackets)

$= 6x^5 + (5x^4 - 5x^4) + (2x^3 - x^3) + (-x^2 - 4x^2) + (4x + 3x) + (-3 + 3)$ (Grouping like terms)

$= 6x^5 + 0x^4 + 1x^3 - 5x^2 + 7x + 0$ (Adding like terms)

$= 6x^5 + x^3 - 5x^2 + 7x$

## Exercise 2.5

1. In each of the following find $f(x) - g(x)$.
   a. $f(x) = -x^3 - 5x^2 + 3x + 12$ and $g(x) = 9x^3 - x^2 - 6x + 3$.
   b. $f(x) = -\frac{1}{3}x^4 + 3x^3 - 3x^2 - \frac{3}{5}x + 2$ and $g(x) = \frac{1}{3}x^4 - x^3 - 2x^2 + \frac{7}{5}x + 5$
   c. $f(x) = -5x^5 + 3x^4 - x^3 + 2x^2 + 5x + 1$ and $g(x) = 2 + 4x - 5x^5 - 3x^4$.
   d. $f(x) = 3\sqrt{3}x^4 + 2x^3 - 5x^2 + x - 5\sqrt{3}$ and $g(x) = \sqrt{3} - 3x - x^2 + 2x^3 - 2\sqrt{3}x^4$.
2. The degree of $f(x) - g(x)$ is equal to the degree of $f(x)$ or the degree of $g(x)$ whichever has the highest degree. (True/False)
3. Is there a possibility for the degree of $f(x) - g(x)$ to be lower than the degree of $f(x)$ or the degree of $g(x)$? When?
4. Is the difference of two polynomial functions a polynomial function?

## Multiplication of Polynomial Functions

### Definition 2.5

The product of two polynomial functions $f(x)$ and $g(x)$ is written as $f \cdot g$, and is defined as:

$$(f \cdot g)(x) = f(x) \cdot g(x) \text{ for all real numbers } x.$$

The product of two polynomials $f(x)$ and $g(x)$ is found by multiplying each term of one by every term of the other as shown in the following example.

### Example 3

1. Find $f(x) \cdot g(x)$ where $f(x) = 2x + 3$ and $g(x) = 3x^2 - 5x + 6$.
2. Let $f(x) = x^2 - 2x + 2$ and $g(x) = 2x^3 - 4x^2 - 5x + 1$ then
   a. Find $f(x) \cdot g(x)$.
   b. Find the degree of $f$, $g$ and $f \cdot g$.

**c.** Is the degree of $f \cdot g$ equal to the sum of the degrees of $f$ and $g$?

**Solution:**

**1.** $f(x) \cdot g(x) = (2x+3)(3x^2-5x+6)$

$= 2x(3x^2-5x+6) + 3(3x^2-5x+6)$ (Distributive property)

$= (2x)(3x^2) + (2x)(-5x) + (2x)(6) + (3)(3x^2) + (3)(-5x) + (3)(6)$

(Distributive property)

$= 6x^3 - 10x^2 + 12x + 9x^2 - 15x + 18$

$= 6x^3 + (-10x^2 + 9x^2) + (12x - 15x) + 18$

$= 6x^3 - x^2 - 3x + 18$

**2. a.** $f(x) \cdot g(x) = (x^2-2x+2)(2x^3-4x^2-5x+1)$

$= x^2(2x^3-4x^2-5x+1) + (-2x)(2x^3-4x^2-5x+1) + 2(2x^3-4x^2-5x+1)$ (Distributive property)

$= 2x^5 - 4x^4 - 5x^3 + x^2 - 4x^4 + 8x^3 + 10x^2 - 2x + 4x^3 - 8x^2 - 10x + 2$ (Distributive property)

$= 2x^5 + (-4x^4 - 4x^4) + (-5x^3 + 8x^3 + 4x^3) + (x^2 + 10x^2 - 8x^2) + (-2x - 10x) + 2$

$= 2x^5 - 8x^4 + 7x^3 + 3x^2 - 12x + 2$

**b.** Degree of $f$ is 2, degree of $g$ is 3 and degree of $f \cdot g$ is 5.

**c.** Yes.

## Exercise 2.6

1. In each of the following find $f(x) \cdot g(x)$.
   a. $f(x) = 3x+1$ and $g(x) = 2x^2+4x-5$
   b. $f(x) = x^2+x-2$ and $g(x) = 3x^2-6x+1$
   c. $f(x) = x^3+3x$ and $g(x) = 2x-x^2$
2. Let $f(x) = 3x^4+2x-4$, $g(x) = x^2-2x^3$.
   a. Find $f(x) \cdot g(x)$.
   b. Find the degree of $f$, $g$ and $f \cdot g$.

c. Is the degree of $f \cdot g$ equal to the sum of the degrees of $f$ and $g$?

To find the product of two polynomial functions, we can also use a vertical arrangement for multiplication.

### Example 4

Find $(x) \cdot g(x)$ , if $f(x) = x^5 + 2x^4 - 3x^2 - x - 5,\ g(x) = 3x^2 - 4x + 2$.

**Solution:**

We arrange the polynomials in a vertical column and multiply each term of the second polynomial by each term of the first polynomial as indicated along with the following solution.

$$\begin{array}{rl} x^5 + 2x^4 - 3x^2 - x - 5 & \\ 3x^2 - 4x + 2 & \\ \hline 2x^5 + 4x^4 \qquad - 6x^2 - 2x - 10 & \text{(Multiplying by 2)} \\ -4x^6 - 8x^5 \qquad + 12x^3 + 4x^2 + 20x & \text{(Multiplying by } -4x\text{)} \\ 3x^7 + 6x^6 \qquad - 9x^4 - 3x^3 - 15x^2 & \text{(Multiplying by } 3x^2\text{)} \\ \hline \end{array}$$

$3x^7 + 2x^6 - 6x^5 - 5x^4 + 9x^3 - 17x^2 + 18x - 10$ (Adding like terms vertically)

Thus, $f(x).g(x) = 3x^7 + 2x^6 - 6x^5 - 5x^4 + 9x^3 - 17x^2 + 18x - 10$.

## Exercise 2.7

Find the product of $f(x)$ and $g(x)$ using vertical arrangement.

a. $f(x) = 2x^2 - 2x - 1$ and $g(x) = 3x + 5$.

b. $f(x) = 3x^3 - x^2 + x - 1$ and $g(x) = 5x - 2x^2$.

c. $f(x) = x^3 + 2x^2 + x - 5$ and $g(x) = -2x^2 + 5x - 3$.

## Observation

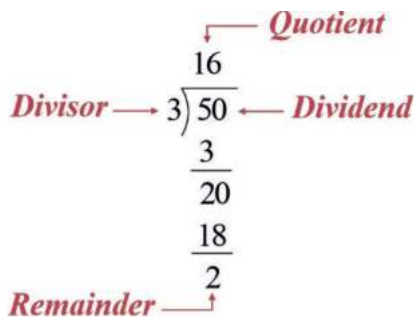
1. For any two non-zero polynomial functions $f$ and $g$, the degree of $f \cdot g$ is $m + n$ if the degree of $f$ is $m$ and the degree of $g$ is $n$.

2. If either $f$ or $g$ is the zero polynomial then $f \cdot g$ becomes the zero polynomial and has no degree.
3. The product of two polynomial functions is a polynomial function.

## Division of Polynomial Functions

A number that takes the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$ is called a rational number. If $b$ is positive integer, we can divide $a$ by $b$ to find two other integers $q$ and $r$ with $0 \leq r < b$ such that $\frac{a}{b} = q + \frac{r}{b}$. Here, $a$ is called the **dividend**, $b$ is called the **divisor**, $q$ is called the **quotient** and $r$ is called the **remainder**.

For example, to find $q$ and $r$ when 50 is divided by 3, you usually use a process called long division as follows:




Hence, $50 \div 3 = \frac{50}{3} = 16 + \frac{2}{3}$.

Here, 50 is the dividend, 3 is the divisor, 16 is the quotient and 2 is the remainder.

In almost a similar way, we can divide one polynomial by another polynomial.

### Activity 2.3

For each of the following, divide the number $a$ by the number $b$ to find two numbers $q$ (quotient) and $r$ (remainder) with $r < b$ such that $\frac{a}{b} = q + \frac{r}{b}$ if

i. $a = 97$ and $b = 8$.
ii. $a = 168$ and $b = 5$.
iii. $a = 287$ and $b = 15$.
iv. $a = 355$ and $b = 11$.

### Definition 2.6

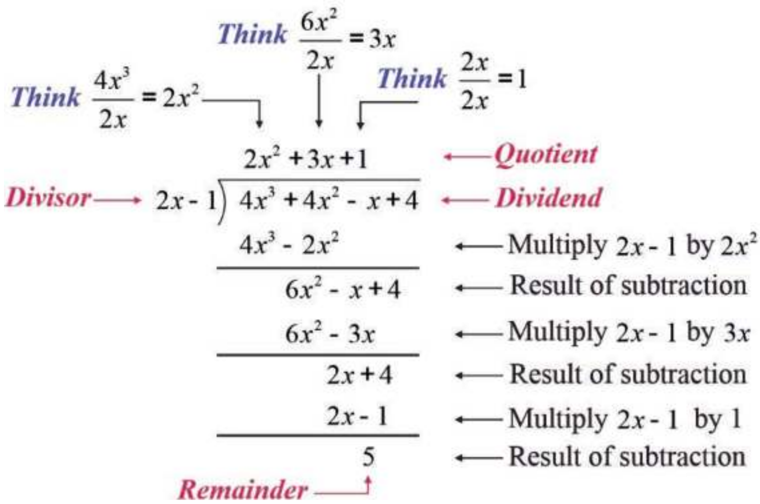
The division (Quotient) of two polynomial functions $f$ and $g$ is written as $f \div g$, and is defined as $(f \div g)(x) = f(x) \div g(x)$ for all real numbers $x$ and $g(x) \neq 0$(zero polynomial).

It is possible to divide one polynomial by another using a **long division process**. When you are asked to divide one polynomial function by another, stop the division process when you get a quotient and remainder that are polynomial functions and the degree of the remainder polynomial is less than the degree of the divisor polynomial.

### Example 5

Find $(x) \div g(x)$ , where $f(x) = 4x^3 + 4x^2 - x + 4$ and $g(x) = 2x - 1$.

**Solution:**




So, dividing $4x^3 + 4x^2 - x + 4$ by $2x - 1$ gives a quotient $2x^2 + 3x + 1$ and a remainder 5.

## Exercise 2.8

In each of the following, find the quotient $q(x)$ and the remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

a. $f(x) = x^3 + 4x^2 + x + 1$ and $g(x) = x + 2$.

b. $f(x) = 4x^3 + 6x^2 - 8x + 5$ and $g(x) = 2x - 1$.

c. $f(x) = x^3 - 3x^2 - 6$ and $g(x) = -x + 1$.

## Example 6

Find $(x) \div g(x)$ , where $f(x) = x^5 + 5x^3 - 2x^2 + 7$ and $g(x) = x^2 + 2x + 1$.

**Solution:**

$$
\begin{array}{r|l}
 & x^3 - 2x^2 + 8x - 16 \\
\hline
x^2 + 2x + 1 & x^5 + 0x^4 + 5x^3 - 2x^2 + 0x + 7 \\
 & x^5 + 2x^4 + x^3 \\
\hline
 & -2x^4 + 4x^3 - 2x^2 + 0x + 7 \\
 & -2x^4 - 4x^3 - 2x^2 \\
\hline
 & 8x^3 + 0x^2 + 0x + 7 \\
 & 8x^3 + 16x^2 + 8x \\
\hline
 & -16x^2 - 8x + 7 \\
 & -16x^2 - 32x - 16 \\
\hline
 & 24x + 23
\end{array}
$$

**Arrange the divisor and the dividend in decreasing power of x**

**Insert 0 (zero) coefficient for missing terms**

**At each step divide the first term of the dividend by the first term of the divisor**

**Multiply the divisor by the result, line up like terms and subtract**

So, dividing $x^5 + 5x^3 - 2x^2 + 7$ by $x^2 + 2x + 1$ gives a quotient $x^3 - 2x^2 + 8x - 16$ and a remainder $24x + 23$.

Therefore, $\frac{x^5+5x^3-2x^2+7}{x^2+2x+1} = x^3 - 2x^2 + 8x - 16 + \frac{24x+23}{x^2+2x+1}$.

## Exercise 2.9

In each of the following, find the quotient $q(x)$ and the remainder $r(x)$ when $f(x)$ is divided by $g(x)$ and write the result as

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.$$

a. $f(x) = x^3 + 3x^2 + 6x + 5; \quad g(x) = x^2 + x + 2$

b. $f(x) = x^4 + x^3 + x^2 - 6x + 7; \quad g(x) = x^2 - 1$

c. $f(x) = 1 + 8x^2 - 5x^3 + 5x^4 + 2x^5; \quad g(x) = 2x^3 - x^2 + 1$

# 2.3 Theorems on Polynomials

## Polynomial Division Theorem

### Activity 2.4

For each of the following, divide the number $a$ by the number $b$ to find two numbers $q$ (quotient) and $r$ (remainder) with $r < b$ such that $a = qb + r$ if

i. $a = 88$ and $b = 5$.

ii. $a = 305$ and $b = 6$.

iii. $a = 354$ and $b = 17$.

iv. $a = 444$ and $b = 111$.

### Theorem 2.1 Polynomial Division Theorem

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where, $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ divides $f(x)$ exactly.

## Proof:

**Existence of the polynomials $q(x)$ and $r(x)$**

Since $f(x)$ and $d(x)$ are polynomials, long division of $f(x)$ by and $d(x)$ will give a quotient $q(x)$ and remainder $r(x)$, with degree of $r(x)$ < degree of $d(x)$ or $r(x) = 0$.

**To show uniqueness of $q(x)$ and $r(x)$**

To prove that $q(x)$ and $r(x)$ are unique, suppose that $q'(x)$ and $r'(x)$ are polynomials satisfying

$$f(x) = q'(x)d(x) + r'(x) \quad and$$

$$r'(x) = 0 \text{ or degree of } r'(x) < \text{degree of } d(x).$$

Then we would have

$$q(x)d(x) + r(x) = f(x) = q'(x)d(x) + r'(x)$$

This implies $$q(x)d(x) + r(x) = q'(x)d(x) + r'(x)$$

$$d(x)(q(x) - q'(x)) = r'(x) - r(x) \ . \ . \ . \ (*)$$

If $q(x) - q'(x) \neq 0$, then the degree of the polynomial on the left-hand side of (*) is greater than or equal to the degree of $d(x)$. But since the polynomials $r'(x)$ and $r(x)$ are either zero or have degree strictly less than that of $d(x)$, the right-hand side of (*) must have degree strictly less than that of $d(x)$. Thus, unless $q(x) - q'(x) = 0$ the degree of the two sides of (*) cannot be the same; that is, we have a contradiction. Therefore, $q(x) - q'(x) = 0$ or $q(x) = q'(x)$.

This implies the left-hand side of (*) is zero. That is, $0 = r'(x) - r(x)$ or $r'(x) = r(x)$. Thus, the polynomials $q'(x)$ and $r'(x)$ are unique.

**Note**

If the remainder $r(x)$ is zero then $d(x)$ divides $f(x)$ exactly and we say the division is exact.

### Example 1

For each of the following pairs of polynomials, find polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$.

**a.** $f(x) = x^3 - 1\ ;\ d(x) = x - 1.$

**b.** $f(x) = -2x^3 + x^2 - 3x + 7\ ;\ d(x) = x^2 - 1.$

**Solution:**

**a.**

$$\frac{x^3}{x} \rightarrow x^2 \qquad \frac{x^2}{x} \rightarrow x \qquad \frac{x}{x} \rightarrow 1$$

$$
\begin{array}{rll}
 & x^2 + x + 1 & \\
x - 1 \,\big) & x^3 \qquad\quad - 1 & \\
 & \underline{x^3 - x^2} & \longleftarrow (x^2)(x-1) \\
 & x^2 \quad - 1 & \longleftarrow (x^3-1)-(x^3-x^2) \\
 & \underline{x^2 - x} & \longleftarrow x(x-1) \\
 & x - 1 & \longleftarrow (x^2-1)-(x^2-x) \\
 & \underline{x - 1} & \longleftarrow (1)(x-1) \\
 & 0 & \longleftarrow (x-1)-(x-1)
\end{array}
$$

$q(x) = x^2 + x + 1$ and $r(x) = 0$. Since $r(x) = 0$ we say that $x^3 - 1$ is exactly divisible by $x - 1$ and $x^3 - 1 = (x^2 + x + 1)(x - 1) + 0 = (x^2 + x + 1)(x - 1)$.

**b.**

$$
\begin{array}{rl}
 & -2x + 1 \\
x^2 - 1 \,\big) & -2x^3 + x^2 - 3x + 7 \\
 & \underline{-2x^3 \qquad + 2x} \\
 & x^2 - 5x + 7 \\
 & \underline{x^2 - 1} \\
 & -5x + 8
\end{array}
$$

The quotient is $q(x) = -2x + 1$ and the remainder is $r(x) = -5x + 8$ such that $-2x^3 + x^2 - 3x + 7 = (-2x + 1)(x^2 - 1) + (-5x + 8)$.

## Exercise 2.10

For each of the following pairs of polynomials, find the quotient $q(x)$ and the remainder $r(x)$ that satisfy the polynomial division theorem.

a. $f(x) = 6x^2 - 2x + 3;\ \ d(x) = x - 1.$

b. $f(x) = x^3 + 4x^2 + 8x + 6;\ \ d(x) = x^2 + 2x - 1.$

c. $f(x) = x^4 + 6x^3 - 10x + 3;\ d(x) = x^2 - 1$

d. $f(x) = -x^3 + 4x^2 - x - 6;\ d(x) = x^2 + x + 1.$

e. $f(x) = -x^4;\ d(x) = x + 2.$

## Remainder Theorem

### Activity 2.5

Find the remainder when the polynomial $f(x)$ is divided by the polynomial $x - c$ for the given number $c$. Compare the result obtained with $f(c)$.

a. $f(x) = x^2 - x + 3;\ c = -2$

b. $f(x) = x^3 + 2x^2 - x - 5;\ c = 1$

### Theorem 2.2 Remainder Theorem

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let $c$ be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

### Proof:

When $f(x)$ is divided by $x - c$ , the remainder is always a constant. Because we continue to divide until the degree of the remainder is less than the degree of the divisor.

By the polynomial division theorem,

$$f(x) = (x - c)q(x) + k,$$

where $k$ is a constant. This equation holds for every real number $x$. Thus, it holds when $x = c$.

That is

$$f(c) = (c - c)q(x) + k$$

$$= 0.q(c) + k = 0 + k = k.$$

It follows that the value of the polynomial $f(x)$ at $x = c$ is the same as the remainder $k$ obtained when you divide $f(x)$ by $x - c$.

### Example 2

In each of the following pairs of polynomials, use remainder theorem to find the remainder when $f(x)$ is divided by $d(x)$.

**a.** $f(x) = 2x^3 + 5x^2 + 3x + 2;\ d(x) = x + 1.$

**b.** $f(x) = x^4 + 3\,;d(x) = x - 2.$

**c.** $f(x) = x^{35} + 5x^{24} - x^9 + 9;\ d(x) = x - 1.$

**d.** $f(x) = 55x^{201} + 100;\ d(x) = x + 1.$

**Solution:**

**a.** $d(x) = x + 1 = x - (-1)$. Therefore, $c = -1$ and the remainder is $f(c) = f(-1) = 2.$

**b.** $d(x) = x - 2,$ therefore $c = 2$ and the remainder is $f(c) = f(2) = 19.$

**c.** $d(x) = x - 1$, therefore $c = 1$ and the remainder is $f(c) = f(1) = 1^{35} + 5(1)^{24} - 1^9 + 9 = 14$

**d.** $d(x) = x + 1 = x - (-1)$, therefore $c = -1$ and the remainder is $f(-1) = 55(-1)^{201} + 100 = 45.$

## Exercise 2.11

In each of the following, use the remainder theorem to find the remainder when $f(x)$ is divided by $d(x)$.

a. $f(x) = x^3 - 3x^2 + 4,\ d(x) = x - 1$

b. $f(x) = -2x^3 + 4x^2 + 5x - 2,\ \ d(x) = x + 2$

c. $f(x) = x^{17} - 4x^2 + 7x - 32,\ \ d(x) = x - 1$

d. $f(x) = x^{16} + 8x^3 + 99,\ \ d(x) = x + 1$

e. $f(x) = x^3 - 2,\ \ d(x) = x - \frac{1}{2}$

## Example 3

**a.** When $3x^3 - 4x^2 + b - 5$ is divided by $x - 2$, the remainder is 10. Find the value of $b$.

**b.** Find the value of $a$ and $b$ such that when $x^3 + ax^2 + bx - 9$ is divided by $x + 1$ and $x - 2$ the remainder is 4 and -5 respectively.

**Solution:**

**a.** Let $f(x) = 3x^3 - 4x^2 + b - 5$, by the remainder theorem when $f(x)$ is divided by $x - 2$ the remainder is $f(2) = 3(2)^3 - 4(2)^2 + b - 5 = b + 3$. Since the remainder is given as 10, we have $b + 3 = 10$ and solving for $b$, we have $b = 7$.

**b.** Let $f(x) = x^3 + ax^2 + bx - 9$. When $f(x)$ is divided by $x + 1$ the remainder is $f(-1) = (-1)^3 + a(-1)^2 + b(-1) - 9 = a - b - 10$.

Since the remainder is 4, $a - b - 10 = 4$,

$$a - b = 14 \quad . \ . \ . \ (1).$$

When $f(x)$ is divided by $x - 2$ the remainder is

$f(2) = (2)^3 + a(2)^2 + b(2) - 9 = 4a + 2b - 1.$

Since the remainder is -5, $4a + 2b - 1 = -5$,

$$4a + 2b = -4$$

$$2a + b = -2 \quad . \ . \ . \ (2).$$

From (1) and (2) we have $a = 4$ and $b = -10$.

## Exercise 2.12

1. When $5x^3 - bx^2 + 8x - 1$ is divided by $x + 1$, the remainder is 15. Find the value of $b$.
2. Find the values of $a$ and $b$ such that when $ax^3 - bx^2 + 5x - 2$ is divided by $x - 1$ and $x + 1$ the remainder is 4 and 6, respectively.

## Factor Theorem

Remember that in the case of multiplication of polynomials, we multiply two or more polynomials to find another polynomial.

For example, $(x+1)(2x-1) = 2x^2 + x - 1$.

The polynomial $2x^2 + x - 1$ is called product or multiple and $(x+1)$ and $(2x-1)$ are called **factors**. Factoring a polynomial means writing it as the product of its polynomial factors. The following theorem is known as the **factor theorem**. It is very helpful to check whether a linear polynomial is a factor of a given polynomial or not.

### Activity 2.6

Let $f(x) = x^3 + 4x^2 + x - 6$.

a. Find $f(1)$.

b. Find the quotient $q(x)$ and the remainder $r(x)$ when $f(x)$ is divided by $x - 1$.

c. Express $f(x)$ as $f(x) = (x-1)q(x) + r(x)$.

d. Is $(x-1)$ a factor of $f(x)$?

### Theorem 2.3 Factor Theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let $c$ be any real number, then

1. if $f(c) = 0$ then $x - c$ is a factor of $f(x)$.
2. if $x - c$ is a factor of $f(x)$ then $f(c) = 0$.

### Proof:

**1.** Suppose $c$ is a real number and $f(c) = 0$ and consider the polynomial $x - c$.

By the polynomial division theorem when a polynomial $f(x)$ is divided by $x - c$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = (x-c)q(x) + r(x),$$

where $r(x) = 0$ or the degree of $r(x)$ is less than "the degree of $x - c$" $= 1$.

But, degree of $r(x) < 1$, means "degree of $r(x)$" $= 0$ that is $r(x)$ is a constant polynomial.

Let $r(x) = k$, since $f(c) = (c - c)q(c) + r(c) = 0,\ r(c) = 0 = k$.

Thus, $r(x) = 0$ and this implies $f(x) = (x - c)q(x)$, thus $x - c$ is a factor of $f(x)$.

**2.** Suppose $c$ is a real number and $x - c$ is a factor of $f(x)$, then there is a polynomial $q(x)$ such that $f(x) = (x - c)q(x)$. From this $f(c) = (c - c)q(x) = 0$.

## Example 4

**a.** Show that $x + 2$ is a factor of $f(x) = x^2 + 5x + 6$.

**b.** Show that $x + 1$ and $x - 2$ are factors but $x + 2$ is not factor of $f(x) = x^4 - x^3 - x^2 - x - 2$.

**Solution:**

**a.** Since $x + 2 = x - (-2)$ has the form $x - c$, the value of $c = -2$.

Now, $f(c) = f(-2) = (-2)^2 + 5(-2) + 6 = 0$, then by the factor theorem $x + 2$ is a factor of $x^2 + 5x + 6$.

**b.** Since $x + 1 = x - (-1)$ has the form $x - c$, the value of $c = -1$.

$f(c) = f(-1) = (-1)^4 - (-1)^3 - (-1)^2 - (-1) - 2 = 0$. Then by the factor theorem $x + 1$ is a factor of $f(x)$.

Since $x - 2$ has the form $x - c$, the value of $c = 2$.

$f(c) = f(2) = (2)^4 - (2)^3 - (2)^2 - (2) - 2 = 0$. Then by the factor theorem $x - 2$ is a factor of $f(x)$.

Since $x + 2 = x - (-2)$ has the form $x - c$, the value of $c = -2$.

$f(c) = f(-2) = (-2)^4 - (-2)^3 - (-2)^2 - (-2) - 2 = 20 \neq 0$. Then by the factor theorem $x + 2$ is not a factor of $f(x)$.

## Exercise 2.13

1. Show that $x - 2$ is a factor of $f(x) = x^2 + x - 6$.
2. Which of the following is a factor of $f(x) = x^3 + 4x^2 + x - 6$?

   a. $x - 1$ b. $x - 3$ c. $x + 1$ d. $x + 2$
3. In each of the following, determine whether $x - c$ is a factor of $f(x)$.

   a. $f(x) = x^3 - 6x^2 + 11x - 6;\ \ c = 1.$

   b. $f(x) = 2x^4 - x^3 + 3x^2 - 4x - 3;\ \ c = -\frac{1}{2}.$

   c. $f(x) = x^3 - 3x^2 + 4x - 3;\ \ c = 2.$

### Example 5

**a.** Find the number $k$ such that $x + 1$ is a factor of $kx^3 + 2x^2 - 3kx + 2$.

**b.** Find the values of $a$ and $b$ in the polynomial $x^3 + ax^2 + bx + 6$ such that $x + 1$ and $x - 2$ are its factors.

**Solution:**

**a.** Let $f(x) = kx^3 + 2x^2 - 3kx + 2$.

$x + 1 = x - (-1)$ is a factor of $f$ implies $f(-1) = 0$.

That is, $f(-1) = 0$

$$k(-1)^3 + 2(-1)^2 - 3k(-1) + 2 = 0$$

$$-k + 2 + 3k + 2 = 0$$

$$2k + 4 = 0$$

$$k = -2$$

**b.** Let $f(x) = x^3 + ax^2 + bx + 6$.

$x + 1$ is a factor of $f(x)$ implies $f(-1) = 0$.

On the other hand,

$$f(-1) = (-1)^3 + a(-1)^2 + b(-1) + 6 = a - b + 5.$$

Therefore, $a - b + 5 = 0$

$$a - b = -5 \ . \ . \ . \ (1)$$

$x - 2$ is a factor of $f(x)$ implies $f(2) = 0$.

On the other hand,

$f(2) = (2)^3 + a(2)^2 + b(2) + 6 = 4a + 2b + 14$.

Therefore, $4a + 2b = -14$

$$2a + b = -7 \quad . . . \; (2)$$

From (1) and (2) we have $a = -4$ and $b = 1$.

## Exercise 2.14

1. In each of the following, find a number $k$ satisfying the given condition.
   a. $x - 2$ is a factor of $2x^3 + kx^2 + 5x - 1$.
   b. $x + 3$ is a factor of $x^4 + 2kx^3 - x^2 - 5kx + 6$.
2. Find the values of $a$ and $b$ in the polynomial $ax^4 + x^3 - 2bx^2 - 11x + 6$ such that $x + 1$ and $x - 2$ are its factors.

## 2.4 Zeros of a Polynomial Function

From your grade 9 mathematics lesson, you know how to find the solution or root of linear and quadratic equations.

### Activity 2.7

Find the solution of the following equations.

a. $2x + 3 = -5 + 3x$

b. $\frac{2}{3}x + \frac{5}{4} = 2(x - \frac{1}{8})$

c. $(2x - 3)(4 + 2x) = 0$

d. $x^2 - 5x + 6 = 0$

e. $x^2 + 2x + 1 = 0$

f. $x^2 + 4 = 0$

In activity 2.7, you have tried to find solutions or roots of the equations.

For a polynomial function $f(x)$, the root of the equation $f(x) = 0$ is called the **zero** of $f(x)$.

## Definition 2.7

For a polynomial function $f(x)$ and a real number $c$, if $f(c) = 0$ then $c$ is a **zero** of $f$.

**Note**

1. If $x - c$ is a factor of $f(x)$, then $c$ is the zero of $f(x)$.
2. If $c$ is the zero of $f(x)$, then $x - c$ is a factor of $f(x)$.
3. If $c$ is the zero of $f(x)$, then $c$ is the root or solution of the equation $f(x) = 0$.

### Example 1

**a.** Find the zeros of $f(x) = (x - 2)(3x + 4)(1 - 4x)$.

**b.** Determine the zeros of $f(x) = x^4 - 13x^2 + 36$.

**c.** If $x = 1$ is one zero of $f(x) = x^3 - 6x^2 + 11x - 6$, then find the rest of real zeros and rewrite $f(x)$ as a product of its factors.

**Solution:**

**a.** $f(x) = 0$

$(x - 2)(3x + 4)(1 - 4x) = 0$

$x - 2 = 0$ or $3x + 4 = 0$ or $1 - 4x = 0$

$x = 2$ or $x = -\frac{4}{3}$ or $x = \frac{1}{4}$

Therefore, $x = 2$, $x = -\frac{4}{3}$ and $x = \frac{1}{4}$ are the zeros of $f(x)$.

**b.** $f(x) = 0$

$x^4 - 13x^2 + 36 = 0$

$(x^2)^2 - 13x^2 + 36 = 0$ (Let $y = x^2$)

$y^2 - 13y + 36 = 0$

$(y - 4)(y - 9) = 0$

$(x^2 - 4)(x^2 - 9) = 0$

$(x - 2)(x + 2)(x - 3)(x + 3) = 0$

$x = 2$ or $x = -2$ or $x = 3$ or $x = -3$

Therefore, $x = 2$ , $x = -2$ , $x = 3$ and $x = -3$ are the zeros of $f(x)$.

**c.** Since $x = 1$ is a zero of $f(x)$, by the factor theorem $x - 1$ is one factor of $f(x)$.

Using long division, the other factor is $x^2 - 5x + 6$.

Further factoring $x^2 - 5x + 6 = (x - 2)(x - 3)$. Hence, $x^3 - 6x^2 + 11x - 6 = 0$ is the same as $(x - 1)(x - 2)(x - 3) = 0$.

Therefore, $x = 1$ , $x = 2$ and $x = 3$ are the zeros of $f(x)$.

## Exercise 2.15

Find the zeros of the following functions.

a. $f(x) = (x - 1)(x + 5)(3x - 2)$

b. $f(x) = x^4 - 5x^2 + 4$

c. $f(x) = x^4 - x^2 - 2$

d. $f(x) = x^3 - x^2 - 10x - 8$

e. $f(x) = 2x^3 - 9x^2 - 5x$

## Zeros of a Polynomial Function and Their Multiplicities

Consider $f(x) = x^3 + 2x^2 - x - 2 = (x - 1)(x + 1)(x + 2)$ and

$g(x) = x^3 + 4x^2 - 3x - 18 = (x + 3)^2(x - 2)$

$f(x)$ has three distinct factors $(x - 1), (x + 1)$ and $(x + 2)$. That is, it has three distinct zeros $1, -1$ and $-2$. These are called **simple zeros** of $f(x)$. While, in $g(x)$ the factor $(x + 3)$ is repeated twice, that is, the zero $-3$ of $g(x)$ is repeated twice and its other zero 2 appears only once. In this case we say -3 is **a repeated or a multiple zero** of $g(x)$.

### Definition 2.8

If $(x - c)^k$ is a factor of a polynomial function $f(x)$, but $(x - c)^{k+1}$ is not, then $c$ is said to be **a zero of multiplicity** $k$ of $f(x)$.

## Example 2

**a.** Given that $x = -1$ is a zero of $f(x) = x^3 - x^2 - 5x - 3$, find its multiplicity.

**b.** Find a polynomial function $f(x)$ of degree two whose zeros are 1, $-2$ and satisfying the condition $f(3) = 30$.

**Solution:**

**a.** By the factor theorem $x + 1$ is a factor of $f(x)$. To find the other factor, you can use long division giving $f(x) = (x + 1)(x^2 - 2x - 3)$. Further factoring the quadratic factor gives $(x + 1)(x + 1)(x - 3) = (x + 1)^2(x - 3)$. Therefore, -1 is a zero of multiplicity 2 of $f(x)$.

**b.** Let $f(x) = k(x - 1)(x + 2)$ for some number $k$. Clearly 1 and $-2$ are zeros of $f(x)$. To find $k$, since $f(3) = k(3 - 1)(3 + 2) = k(2)(5) = 10k = 30$, implies $k = 3$. Therefore, the polynomial function of degree two is

$f(x) = 3(x - 1)(x + 2) = 3x^2 + 3x - 6$

## Exercise 2.16

1. Given that $x = 1$ is a zero of $f(x) = x^3 - 4x^2 + 5x - 2$, find the other zeros and their multiplicity.
2. For each of the following polynomials, list the zeros and state the multiplicity of each zero.
   a. $k(t) = (t + \frac{2}{3})^3 t^{10}$.
   b. $g(x) = 5(x + \xi \overline{2})^2 (x + 2)^3 (1 + 3x)$.
   c. $h(t) = 2t^3 + 5t^2 + 4t + 1$.
3. a. Find a polynomial $f(x)$ of degree two whose zeros are -2, 3 and satisfying the condition $f(2) = 12$.
   b. Find a polynomial function $f(x)$ of degree three whose zeros are $-1$, 2 and 1 and satisfying the condition $f(3) = 16$.
   c. Find a polynomial function $f(x)$ of degree 7 such that 2, $-3$, and 0 are the zeros of multiplicity 3, 2 and 2, respectively and $f(1) = 48$.

## Location Theorem

### Activity 2.8

In each of the following, determine whether the zeros of the polynomial function are rational, irrational, or neither.

a. $f(x) = x(x+2)(x-\frac{3}{2})$.  b. $f(x) = (x+\sqrt{2})(x-\sqrt{3})$.

c. $f(x) = x^2 + 1$.

Consider $f(x) = x^2 - 3$. $-\sqrt{3}$ and $\sqrt{3}$ are the zeros of $f$ and they are irrational numbers. The table of values of the given functions for $-2 \leq x \leq 2$ and $x$ is an integer, is the following.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $1$ | $-2$ | $-3$ | $-2$ | $1$ |

Since $f(-2) = 1 > 0$ and $f(-1) = -2 < 0$, we see that the value of $f(x)$ changes sign from positive to negative between $-2$ and $-1$. And observe that one of the irrational roots $-\sqrt{3} \cong -1.73$ lies between these two numbers. We also see that the value of $f(x)$ changes sign from negative to positive between $1$ and $2$. Similarly observe that the second irrational root $\sqrt{3} \cong 1.73$ lies between these two numbers.

The following theorem which is called the location theorem helps to locate the real zeros of a polynomial function.

### Theorem 2.4 Location Theorem

Let $a$ and $b$ be real numbers such that $a < b$. If $f$ is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of $f$ between the numbers $a$ and $b$.

It is sometimes possible to estimate the zeros of a polynomial function from a table of values.

## Example 3

Let $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4$. Construct a table of values and use the location theorem to locate the zeros of $f$ for the integers $x$ and $-3 \leq x \leq 3$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 91 | 12 | $-1$ | 4 | 3 | $-4$ | 7 |

**Solution:**

Since $f(-2) = 12 > 0$ and $f(-1) = -1 < 0$, we see that the value of $f(x)$ changes sign from positive to negative between $-2$ and $-1$. Hence by the location theorem there is a zero of $f(x)$ between $x = -2$ and $x = -1$.

Since $f(-1) = -1 < 0$ and $f(0) = 4 > 0$, we see that the value of $f(x)$ changes sign from negative to positive between $-1$ and $0$. Hence by the location theorem there is a zero of $f(x)$ between $x = -1$ and $x = 0$. Similarly, there are zeros of $f(x)$ between $x = 1$ and $x = 2$ and between $x = 2$ and $x = 3$.

## Example 4

Using the location theorem, show that the polynomial function

$f(x) = x^6 - 2x^5 - 4x^2 + 4x + 4$ has a zero between $x = 1$ and $x = 2$.

**Solution:**

$f(1) = (1)^6 - 2(1)^5 - 4(1)^2 + 4(1) + 4 = 3 > 0$

$f(2) = (2)^6 - 2(2)^5 - 4(2)^2 + 4(2) + 4 = -4 < 0$

Hence, $f(1)$ is positive and $f(2)$ is negative and by the location theorem $f(x)$ has a zero between $x = 1$and $x = 2$.

## Exercise 2.17

1. Use the location theorem to verify that $f(x)$ has a zero between $a$ and $b$.
   a. $f(x) = -x^4 + x^3 + 1\,;\ a = -1\,, b = 1.$

b. $f(x) = 3x^3 + 7x^2 + 3x + 7$ ; $a = -3$ , $b = -2$.

2. In each of the following, use the Location Theorem to locate the real zero of $f(x)$ between successive integers in the given intervals.

   a. $f(x) = x^3 - 9x^2 + 23x - 14$; for $0 \leq x \leq 6$.

   b. $f(x) = x^4 + 2x^3 - 4x^2 - 6x + 2$; for $-3 \leq x \leq 3$.

## Rational Zero Test

**Theorem 2.5 Rational Zero Test**

Suppose that all the coefficients of the polynomial function described by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

are integers with $a_n \neq 0$ and $a_0 \neq 0$. If $\frac{p}{q}$ in lowest term is a zero of $f(x)$, then $p$ is a factor of $a_0$ and $q$ is a factor of $a_n$.

**Steps to find the rational zeros of a polynomial function $f(x)$.**

1. Arrange the polynomial in descending order so that it takes the form
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0.$$
2. Write down all the factors of the constant term $a_0$. These are all the possible values of $p$.
3. Write down all the factors of the leading coefficient $a_n$. These are all the possible values of $q$.
4. Write down all the possible values of $\frac{p}{q}$. Remember that since factors can be negative, $\frac{p}{q}$ and $-\frac{p}{q}$ must both be included. Simplify each value and cross out any duplicates.
5. Identify those values of $\frac{p}{q}$ for which $f\left(\frac{p}{q}\right) = 0$. These are all the rational zeros of $f(x)$.

## Example 5

In each of the following, find all the rational zeros of the polynomial.

**a.** $f(x) = x^2 + x - 2$ **b.** $f(x) = x^3 - x + 1$

**c.** $f(x) = 6x^3 + 13x^2 + x - 2.$ **d.** $f(x) = \frac{1}{2}x^3 + x^2 - \frac{1}{2}x - 1$

**Solution:**

**a.** $f(x) = x^2 + x - 2$ has leading coefficient $a_2 = 1$ and constant term $a_0 = -2$.

Possible values of $p$ are factors of $-2$. These are $\pm 1, \pm 2$.

Possible values of $q$ are factors of 1. These are $\pm 1$.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2$. Since $f(x)$ is a polynomial function of degree 2, it has at most 2 zeros, and from the four possible rational zeros at most 2 can be the zeros of $f$. We can check this using the table below

| $x$ | $-2$ | $-1$ | $1$ | $2$ |
|---|---|---|---|---|
| $f(x)$ | 0 | $-2$ | 0 | 4 |

Therefore, the zeros of $f(x)$ are $-2$ and 1.

**b.** $f(x) = x^3 - x + 1$, the leading coefficient is 1 and the constant term is 1. Hence, the possible rational zeros are $\pm 1$.

Check that $f(1) = f(-1) = 1 \neq 0$. So, we can conclude that the given polynomial has no rational zero. Use the location theorem to show that $f$ has a zero between $-2$ and $-1$.

**c.** The leading coefficient is $a_3 = 6$ and the constant term is $a_0 = -2$.

Possible values of $p$ are factors of $-2$. These are $\pm 1, \pm 2$.

Possible values of $q$ are factors of 6. These are $\pm 1, \pm 2, \pm 3$ and $\pm 6$.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}$. Since $f(x)$ is a polynomial function of degree 3 it has at most three zeros and from the 12 possible rational zeros at most 3 can be the zeros of $f$.

Check that $f\left(-\frac{1}{2}\right) = f(-2) = f\left(\frac{1}{3}\right) = 0$.

Using the factor theorem, we can factorize $f(x)$ as $f(x) = (3x-1)(2x+1)(x+2)$ with $-\frac{1}{2}$, $-2$ and $\frac{1}{3}$ are the only rational zeros of $f$.

**d.** $f(x) = \frac{1}{2}x^3 + x^2 - \frac{1}{2}x - 1 = \frac{1}{2}(x^3 + 2x^2 - x - 2) = \frac{1}{2}k(x)$

where, $k(x) = x^3 + 2x^2 - x - 2$ has integer coefficients and the same zeros as $f(x)$. $k(x)$ has a constant term $-2$ and leading coefficient 1. The possible rational zeros $k(x)$ are $\pm 1$ and $\pm 2$. You can check that $k(1) = k(-1) = k(-2) = 0$. Therefore, the zeros of $f(x)$ are $\pm 1$ and $-2$.

## Exercise 2.18

For each of the following polynomials, find all possible rational zeros:

a. $f(x) = x^2 - 5x + 4$

b. $f(x) = -3x^3 + x^2 - 3x + 1$

c. $f(x) = x^3 - 3x^2 - x - 3$

d. $f(x) = 10x^3 - 41x^2 + 2x + 8$

e. $f(x) = 4x^4 + x^3 - 8x^2 - 18x - 4$

f. $f(x) = -6x^5 + 17x^4 - 14x^3 + 4x - 1$

## 2.5 Graphs of Polynomial Functions

In unit 1, you discussed how to draw the graphs of the polynomial functions of degree zero, one and two. You saw that the graph of a linear function is a straight line and the graph of a quadratic function is a parabola. In this section you will learn about the properties of graphs of polynomial functions.

## Activity 2.9

1. Use table of values to sketch the graphs of $f(x) = 2x + 3$ and $g(x) = -2x + 3$. Include the $x$-intercept and y-intercept when you make the table of values. Sketch the graphs on the same $xy$-plane.
2. Consider the linear function $f(x) = ax + b$, $a \neq 0$ and give answers for each of the following.
   a. What is its degree? Is it odd or even?
   b. Find the intercepts.
   c. Write the behavior of the graph when $a$ is positive and $|x|$ is large (far to the right and far to the left).
   d. Write the behavior of the graph when $a$ is negative and $|x|$ is large (far to the right and far to the left).
   e. What is the shape of the graph of $f$?
   f. Find the domain and the range of $f$.

### Example 1

For the function $f(x) = x^2 + 2x - 3$,

**a.** Find the intercepts.

**b.** Using completing the square method, rewrite $f$ as $f(x) = -4 + (x+1)^2$ and find the turning point.

**c.** Complete the table of values below

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | | | | | | | |

**d.** Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve (a smooth curve is a curve that has no sharp corner).

**e.** Find the domain and range of $f$.

**Solution:**

**a.** $f(x) = y = x^2 + 2x - 3 = (x+3)(x-1)$. By making $f(x) = y = 0$ and solving the equation $x^2 + 2x - 3 = (x+3)(x-1) = 0$, we get $x = -3$ or $x = 1$. By making $x = 0$ we have $y = -3$. Thus, $x = -3$ and $x = 1$ are the $x-$ intercepts and $y = -3$ is the $y$-intercept.

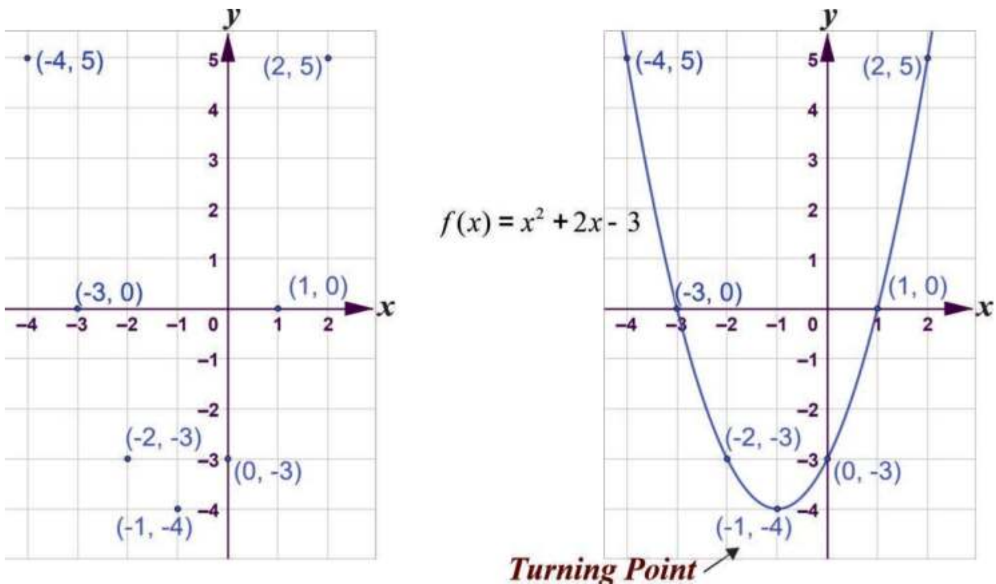
**b.** $y = x^2 + 2x - 3 = (x^2 + 2x + 1) - 1 - 3 = (x+1)^2 - 4 = -4 + (x+1)^2$

Since $(x+1)^2 \geq 0$ for all real numbers $x$, $f(x) \geq -4$ for all values of $x$ and $-4$ is the minimum value of $f$. This minimum value of $f$ is attained when $x = -1$. **The point $(-1, -4)$ is called turning point or vertex of the graph of $f$.**

**c.**

| $x$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | 5 | 0 | -3 | -4 | -3 | 0 | 5 |

**d.**



(a) (b)

Figure 2.1

**e.** The domain is the set of all real numbers and the range is the set of all real numbers greater than or equal to $-4$.

## Exercise 2.19

For the function $f(x) = x^2 + 4x + 3$,

a. Find the intercepts.

b. Using completing the square method, rewrite $f$ as $f(x) = -1 + (x + 2)^2$ and find the turning point.

c. Complete the table of values below

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | | | | | | | |

d. Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve.

e. Find the domain and range of $f$.

### Example 2

Let $f(x) = -x^2 - 6x - 8$.

**a.** Find the intercepts.

**b.** Using completing the square method, rewrite $f$ as $f(x) = 1 - (x + 3)^2$ and find the turning point.

**c.** Complete the table of values below

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|---|
| $y = f(x)$ | | | | | |

**d.** Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve.

**e.** Find the domain and range of $f$.

## Solution:

**a.** $f(x) = y = -x^2 - 6x - 8 = (x+2)(-x-4)$. By making $f(x) = 0$ and solving the equation $-x^2 - 6x - 8 = (x+2)(-x-4) = 0$,we get $x = -2$ or $x = -4$.

By making $x = 0$ we have $y = -8$. Thus, $x = -2$ and $x = -4$ are the $x-$ intercepts and $y = -8$ is the $y$-intercept.

**b.** $y = f(x) = -x^2 - 6x - 8 = -(x^2 + 6x) - 8$

$$= -(x^2 + 6x + 9) + 9 - 8$$
$$= -(x+3)^2 + 1$$
$$= 1 - (x+3)^2$$

Since $(x+3)^2 \geq 0$ , $f(x) \leq 1$ for all values of $x$ and 1 is the maximum value of $f$. This maximum value of $f$ is attained when $x = -3$. The point $(-3, 1)$ is called turning point for the graph of $f$.

**c.**

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|---|
| $y = f(x)$ | $0$ | $1$ | $0$ | $-3$ | $-8$ |

**d.**

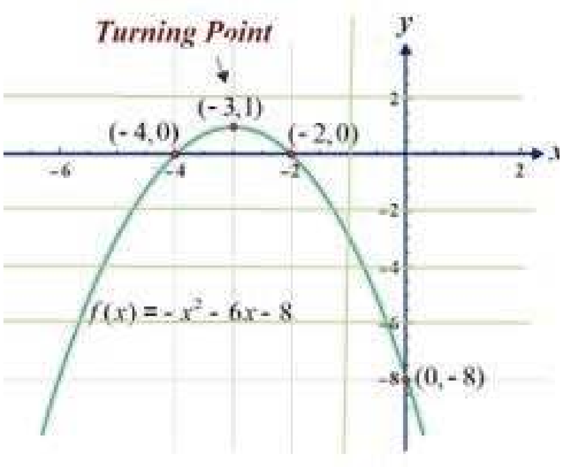



Figure 2.2

**e.** The domain is the set of all real numbers and the range is the set of all real numbers less than or equal to 1.

## Exercise 2.20

1. For the function $f(x) = -x^2 + 6x - 8$,

   a. Find the intercepts.

   b. Using completing the square method, rewrite $f$ as $f(x) = 1 - (x - 3)^2$ and find the turning point.

   c. Complete the table of values below.

| $x$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | | | | | | | |

   d. Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a curve.

   e. Find the domain and range of $f$.

3. Consider the quadratic function $f(x) = ax^2 + bx + c$ and give answers for each of the following.

   a. What is its degree? Is it even or odd?

   b. What is the maximum number of $x$-intercepts?

   c. What is the $y$-intercept?

   d. Write the behavior of the graph when $a$ is positive and àxà is large (far to the right and far to the left)

   e. Write the behavior of the graph when $a$ is negative and àxà is large (far to the right and far to the left)

   f. Is the graph smooth (has no sharp corner) and continuous (has no jump or hole)?

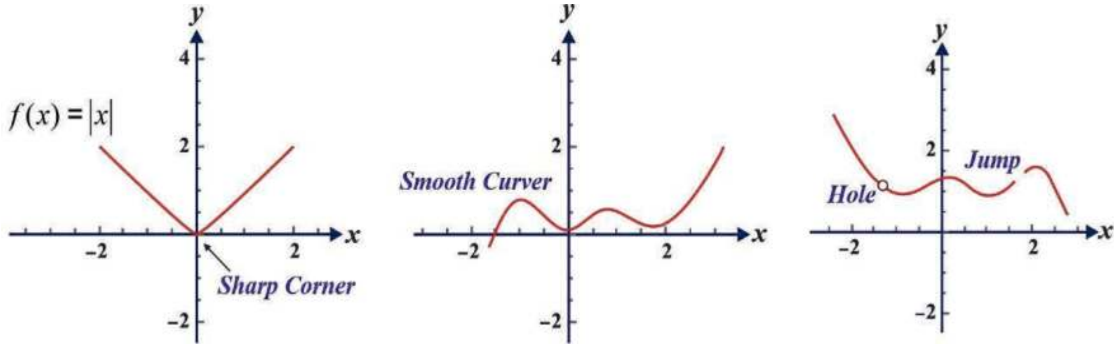
   g. What is the domain of $f$?

   h. Can the range of $f$ be all real numbers? Why?

**Note**

1. Graph of a polynomial function is a smooth curve (has no sharp corner).
2. Functions whose graphs are not continuous and have sharp corners are not polynomial functions.

The absolute value function $f(x) =$ àxà is not a polynomial function. Because it has a sharp corner at the point (0, 0) as shown by figure 2.3 (a).




(a) (b) (c)

Figure 2.3

The function shown in figure 2.3 ($c$) is not polynomial because it is not a continuous function. It has a hole and a jump.

To study more on the property of graphs of polynomials, we will now try to observe the graphs of polynomial functions of higher degree, that is, when the degree $n \geq 3$.

### Example 3

By sketching the graphs of $f(x) = x^3 - 1$ and $g(x) = -x^3 + 1$, describe the behavior of the graphs for large àxà

**Solution:**

For $f(x) = x^3 - 1 = (x-1)(x^2 + x + 1)$, the $x$-intercept is 1, the $y$-intercept is $-1$ and by finding some points that lie on the graph of $f$ as shown by the table of values.

| $x$ | $-20$ | $-10$ | $-1$ | 0 | 1 | 1.5 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|
| $y$ | $-8001$ | $-1001$ | $-2$ | $-1$ | 0 | 2.375 | 999 | 7999 |

We plot the points $(-1,-2)$, $(0,1)$, and $(1.5,\frac{19}{8})$ and connect them by a smooth curve and we use the other points to see the direction of the graph far to the right and far to the left along the $x$- axis.

For $g(x) = -x^3 + 1 = (-x+1)(x^2+x+1)$, the $x$-intercept is 1, the $y$-intercept is 1 by finding some points that lie on the graph of $g$ as shown by the table of values.

| $x$ | $-20$ | $-10$ | $-1$ | $0$ | $1$ | $1.5$ | $10$ | $20$ |
|---|---|---|---|---|---|---|---|---|
| $y$ | $8001$ | $1001$ | $2$ | $1$ | $0$ | $-2.375$ | $-999$ | $-7999$ |

Similarly, we plot the points $(-1,2)$, $(0,1)$ and $(1.5,-\frac{19}{8})$ and connect them by a smooth curve and we use the other points to see the direction of the graph far to the right and far to the left along the $x$-axis. The graphs of the two functions are:

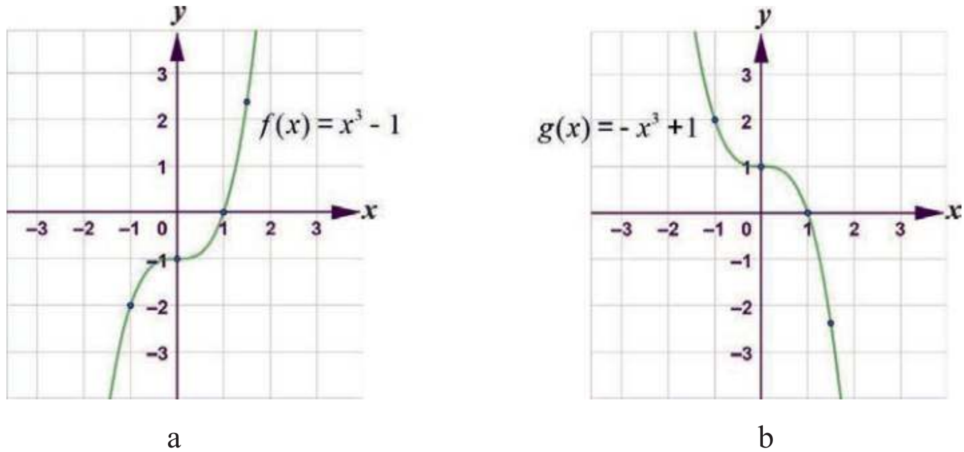



Figure 2.4

**Observation**

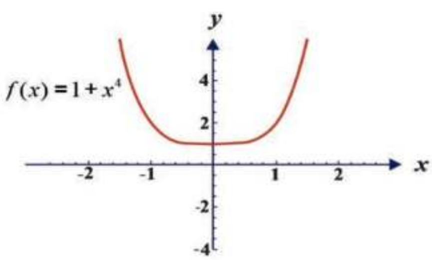
1. The degree of $f(x)$ is odd and its leading coefficient is positive. As shown in figure 2.4a, when $x$ is large positive, $f(x)$ becomes large positive and the graph moves upwards and when $x$ is large negative, $f(x)$ becomes large negative and the graph moves downwards.
2. The degree of $g(x)$ is odd and its leading coefficient is negative. As shown in figure 2.4b, when $x$ is large positive, $f(x)$ becomes large negative and the graph

moves downwards and when $x$ is large negative, $f(x)$ becomes large positive and the graph moves upwards.

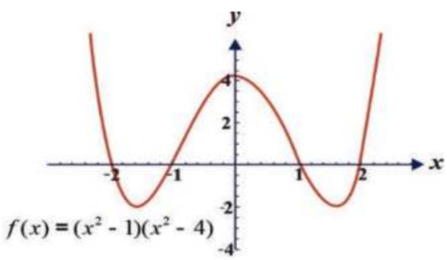
## Exercise 2.21

1. By sketching the graphs of $f(x) = x^3$ and $g(x) = -x^3$, describe the behavior of the graphs for large àxà
2. For the third- degree polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$,
   a. What is the maximum number of intersections which the graph of $f$ makes with the $x$-axis?
   b. What is the minimum number of intersections which the graph of $f$ makes with the $x$-axis?
   c. What is the number of intersections which the graph of $f$ makes with the $y$-axis?
   d. What maximum number of turning points dose the graph of $f$ have?
   e. Write the behavior of the graph when $a_3$ is positive and àxà is large (far to the right and far to the left).
   f. Write the behavior of the graph when $a_3$ is negative and àxà is large (far to the right and far to the left).
   g. What is the domain of $f$?

Noted that the following are examples of graphs of polynomial functions of degree four. The graphs are drawn using graph calculator software called GeoGebra.




a




b

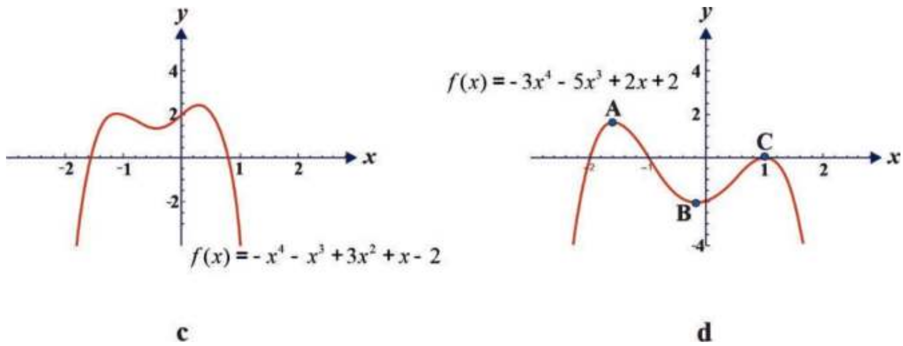



Figure 2.5

In figure 2.5a and figure 2.5b, the leading coefficient of $f(x)$ is positive and the degree is even. The values of $f(x)$ become large positive and the graphs go upward both far to the right and far to the left as the values of $x$ become large in absolute value.

In figure 2.5c and figure 2.5d, the leading coefficient of $f(x)$ is negative and the degree is even. The values of $f(x)$ become large negative and the graphs go downward both far to the right and far to the left as the values $x$ become large in absolute value.

From figure 2.5b, it is seen that the maximum number of intersections that the graph of a fourth-degree polynomial makes with the $x$-axis is 4 and the maximum number of turning points is 3.

As shown in figure 2.5a, the graph of a polynomial function of degree 4 may not intersect the $x$-axis.

By applying the rational root test and the factor theorem, there is a possibility of finding the $x$-intercepts of a function. There is also a possibility of locating the real zeros using the location theorem.

With quadratic polynomials, we were able to algebraically find the maximum or minimum value of the function by finding the vertex (turning point). But for general polynomials, finding the turning points like A, B and C in figure 2.5d, is not possible without more advanced techniques from calculus (derivative of a function).

**Observation**

The properties of the first-degree and the third-degree polynomial are also applicable for polynomial functions of odd degree. The properties of the second-degree and the fourth-degree polynomial are also applicable for polynomial functions of even degree

## Exercise 2.22

1. For the polynomial functions given from $a$ to $d$, state the following properties of the graph of the functions without drawing their graphs.
   - i. The behavior of the graph as $x$ takes values far to the right.
   - ii. The behavior of the graph as $x$ takes values far to the left.
   - iii. The number of intersections with the $x$-axis.

   a. $f(x) = x^2 - 2x - 1$    b. $f(x) = 2x^2 - x^4$

   c. $f(x) = -x^3 + 3x - 2$    d. $f(x) = -2(1-x)^3(x+1)^2$

2. Graphs of some polynomial functions are given below. In each case identify the sign of the leading coefficient of the function. State whether the degree is even or odd.

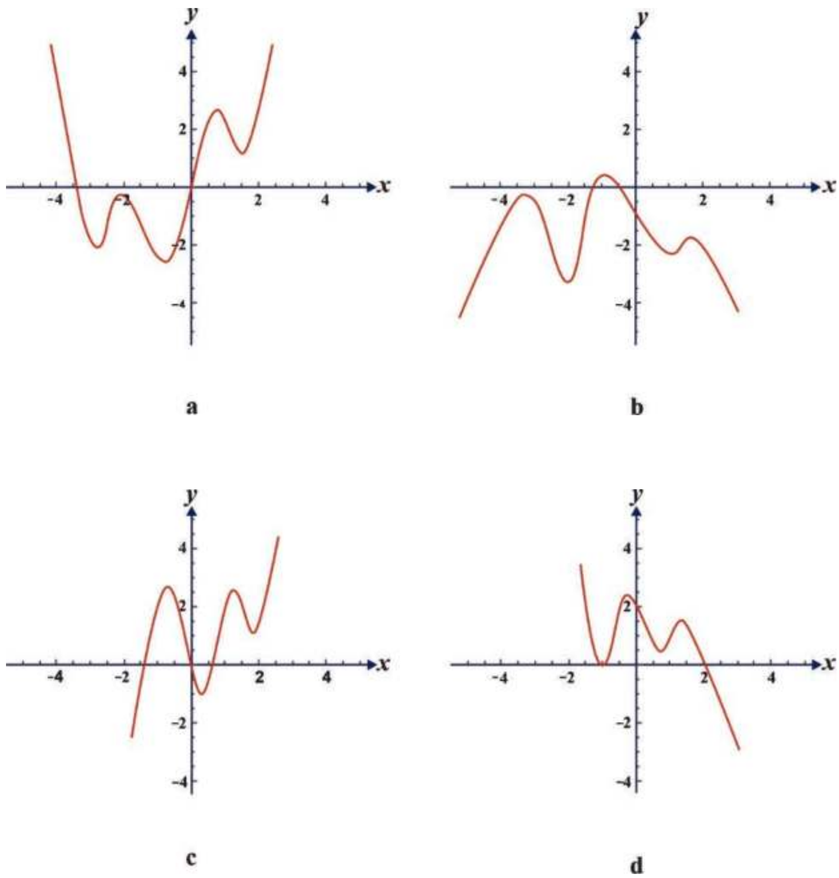



Figure 2.6

## 2.6 Applications

Polynomials arise in the study of problems involving areas and volumes.

### Example 1

A wire of length 56 m is bent into the shape of a rectangle. Find the maximum area it can enclose and the dimensions of the rectangle of maximum area.

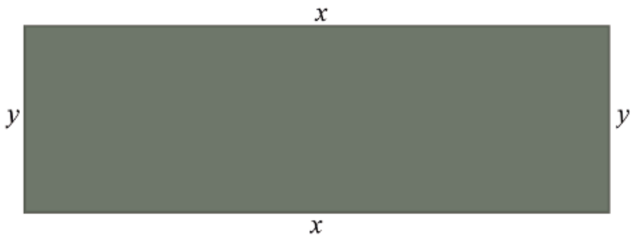



Figure 2.7

**Solution:**

Perimeter $p$ of the rectangle, $p = 2x + 2y \; . \; . \; .$ (i)

Area $A$ of the rectangle, $A = xy \; . \; . \; .$ (ii)

Since the rectangle is made by bending 56 meters wire, the perimeter of the rectangle is 56 m $\quad 2x + 2y = 56 \ldots$ (iii)

From (iii), solving for $y$ (you can also solve for $x$).

$y = 28 - x \; . \; . \; .$ (iv)

Substituting the value of $y$ in (iv) to the value of $y$ in (ii) we get

$A = x(28 - x) = -x^2 + 28x$ , and this is a quadratic polynomial.

Using completing the square method, we can rewrite the value A as in the following:

$\text{A} = -x^2 + 28x = -(x^2 - 28x) = -(x^2 - 28x + 196) + 196 = 196 - (x - 14)^2$

Thus, $\text{A} = 196 - (x - 14)^2$ and since $(x - 14)^2$ is always non-negative, we subtract positive number or zero from 196. This means the maximum value of A is 196. This maximum value is attained when $x$ is 14. Substituting $x = 14$ into equation (iv) we get $y = 14$.

Therefore, the maximum area that can be enclosed is $196\ m^2$ and the dimension of the rectangle of maximum area is $x = 14\ m$ and $y = 14\ m$. That is when the rectangle is a square of side of length 14 m.

## Exercise 2.23

1. A farmer has 100 meters of fencing material to use to make a rectangular enclosure for sheep as shown. He will leave an opening of 2 meters for the gate.

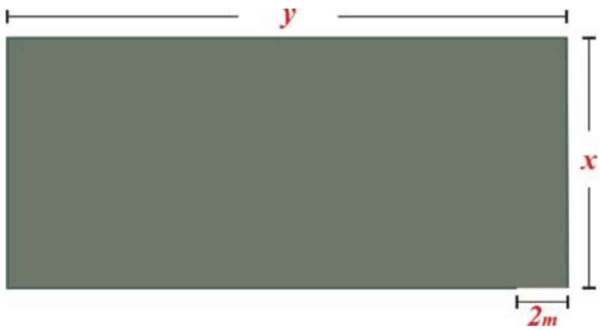



Figure 2.8

   a. Show that the area of the enclosure is given by $A = 51x - x^2$
   b. Find the value of $x$ that will give maximum area.
   c. Calculate the maximum possible area.

2. A farmer has 100 meters of fencing material to make a rectangular enclosure for sheep. One side of the enclosure is closed by a wall as shown. He will leave an opening of 2 meters for the gate.

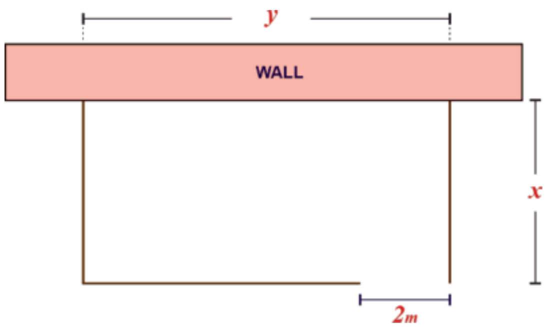



Figure 2.9

   a. Show that the area of the enclosure is given by $A = 102x - 2x^2$
   b. Find the value of $x$ that will give maximum area.
   c. Calculate the maximum possible area.

3. An open-topped box is to be made by removing squares from each corner of a rectangular piece of card and folding up the sides.

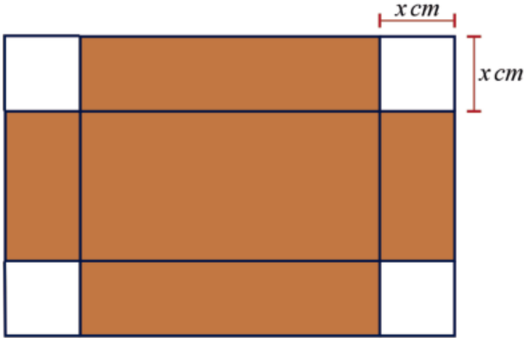



Figure 2.10

a. Show that if the original rectangle of card measured 80 cm by 50 cm and the squares removed from the corners have sides $x$ cm long, then the volume of the box is given by $V = 4x^3 - 260x^2 + 4000x$.

b. Find V when $x = 20$ cm.

c. Can the value of $x$ be 30 cm? Why?

## Example 2

Find the two real numbers whose difference is 16 and whose product is the minimum.

**Solution:**

Let the two numbers be $x$ and $y$. Then,

$x - y = 16$ . . . (1)

If we denote the product of the two numbers by $A$, then

$A = xy$ . . . (2)

To find the minimum value of $A$, first we solve either for $x$ or for $y$ from equation (1).

Solving for $y$ from equation (1) gives $y = x - 16$ and substituting this in equation (2) gives

$$A = x(x - 16) = x^2 - 16x$$
$$= (x^2 - 16x + 64) - 64$$
$$= (x - 8)^2 - 64$$

$$= -64 + (x-8)^2$$

Since $(x-8)^2 \geq 0$

$A = -64 + (x-8)^2 \geq -64$.

Therefore, $-64$ is the minimum product and this minimum product is obtained when the value of $x = 8$. And when $x = 8$, $y = x - 16 = -8$. That is, $y = -8$.

## Exercise 2.24

1. Find the two real numbers whose difference is 10 and whose product is the minimum.
2. Find the two real numbers whose sum is 12 and whose product is the maximum.

## Summary

1. A linear function is given by $f(x) = ax + b,\ a \neq 0$.
2. A quadratic function is given by $f(x) = ax^2 + bx + c,\ a \neq 0$.
3. Let $n$ be a non-negative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$, the function $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called a polynomial function in $x$ of degree n.
4. A polynomial function is over integers if its coefficients are all integers.
5. A polynomial function is over rational numbers if its coefficients are all rational numbers.
6. A polynomial function is over real numbers if its coefficients are all real numbers.
7. Operations on polynomial functions
   i. Sum: $(f + g)(x) = f(x) + g(x)$
   ii. Difference: $(f - g)(x) = f(x) - g(x)$
   iii. Product: $(f \cdot g)(x) = f(x) \cdot g(x)$
   iv. Quotient: $(f \div g)(x) = f(x) \div g(x),\ if\ g(x) \neq 0$
8. If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)d(x) + r(x)$, where $r(x) = 0$ or the degree of $r(x)$ is less that the degree of $d(x)$.
9. If a polynomial $f(x)$ is divided by a first-degree polynomial of the form $x - c$, then the remainder is the number $f(c)$.
10. Given the polynomial function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

    If $p(c) = 0$, then $c$ is a zero of $p(x)$ or a root of the equation $p(x) = 0$.
11. For every polynomial function $f$ and a real number $c$ if $f(c) = 0$, then $x - c$ is a factor of the polynomial function $f$.

**12.** If $(x-c)^k$ is a factor of $f(x)$, but $(x-c)^{k+1}$ is not, we say that $c$ is a zero of multiplicity $k$ of $f(x)$.

**13.** If the rational number $\frac{p}{q}$, in its lowest term, is a zero of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with integer coefficients, then $p$ must be an integer factor of $a_0$ and $q$ must be an integer factor of $a_n$.

**14.** Let $a$ and $b$ be real numbers such that $a < b$. If $f(x)$ is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of $f(x)$ between $a$ and $b$.

**15.** The graph of a polynomial function of degree $n$ has at most $n-1$ turning points and intersects the $x$ -axis at most $n$ times.

**16.** The graph of every polynomial function has no sharp corners; it is a smooth and continuous curve.

## Review Exercise

**1.** Identify whether the following functions are polynomial or not, for those which are polynomial indicate the degree, leading coefficient and constant term.

**a.** $f(x) = 5x^2 - \frac{2}{3}x^3 - \frac{1}{3}x - \frac{4x^3+9x^2-2x+9}{3}$

**b.** $f(x) = 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) - 6$

**c.** $f(x) = 3(x^2)^3 - 4(x^2)^2 + 2(x+1)^2 + 5$

**d.** $f(x) = -2(\xi\,\overline{x})^3 + 5\xi\,\overline{x} - 10$

**e.** $f(x) = 3\pi^2 + 4$

**2.** Given $f(x) = 6 + 4x - 2x^2 + 3x^3$, $g(x) = x^4 - 5x^2 + x^3 - 2$ and $h(x) = x + 2$, find

**a.** $hf + g$ **b.** $f - hg$

**c.** $fg$ **d.** $\frac{g}{h}$

**3.** If $f$ and $g$ are any two polynomials, then which of the following will always be a polynomial function?

**a.** $f + g$ **b.** $f - g$

**c.** $f.g$ **d.** $\frac{f}{g}$

**e.** $f^2 - g$ **f.** $\frac{f+g}{g-f}$

**g.** $3f - \frac{4}{5}g$

**4.** In each of the following, find the quotient and the remainder when the first polynomial is divided by the second:

**a.** $f(x) = 3x^3 + 5x^2 - 7x - 6;\ x + 1$

**b.** $f(x) = 4x^3 - 5x^2 + 4x - 17;\ x^2 - 1$

**c.** $f(x) = 2x^4 + 5x^2 - 6;\ x^2 - x + 1$

**d.** $f(x) = x^5 + 3x^4 + 2x^3 - x^2 + 2x - 7:\ x + 2$

e. $f(x) = x^5 + 2x^4 - x^3 + 5x^2 - x - 2;\ x^3 + 1$

f. $f(x) = 2x^3 - x^2 + 2x - 1:\ 2x + 1$

5. Prove that when a polynomial $f(x)$ is divided by a first-degree polynomial $ax + b$, the remainder is $f(-\frac{b}{a})$.

6. Let $f(x) = x^n + 1$ be polynomial function and $n$ is an odd integer then show that

a. the remainder when $f$ is divided by $x + 1$ is zero.

b. $x + 1$ is a factor of $f$.

7. Factorize fully

a. $x^3 - 4x^2 - 7x + 10$

b. $2x^4 - x^3 - 6x^2 + 7x - 2$

c. $2x^5 + 2x^4 - x^3 - x^2 - x - 1$

8. Find the value of $k$ such that,

a. when $f(x) = 3x^3 - 2x^2 + kx - 6$ is divided by $x - 3$ it has a remainder of -3.

b. $x + 1$ is a factor of $x^3 - kx^2 + 4x - 1$.

c. $2x - 3$ is a factor of $x^3 + 3x^2 + kx - 10$.

9. When the polynomial $f(x) = a(2x + 1)^2 + b(x - 2)^2$ is divided by $x + 1$ the remainder is $-10$ and $f(1) = 10$. Then find the values of $a$ and $b$.

10. Find the values of $p\ and\ q$ if $x + 1$ is a common factor of $f(x) = 2x^4 - px^3 + qx^2 + 2$ and $g(x) = px^5 + 8x^3 - 4x^2 - qx$.

11. Find numbers $a$ and $k$ so that $x + 1$ is a factor of $f(x) = ax^4 - 2kx^3 + ax^2 - kx + 2$ and $f(1) = 2$.

12. Find a polynomial function $f$ of degree 3 such that $f(2) = 48$ and $x + 1$, $x$ and $x + 2$ are factors of the polynomial.

13. In each of the following, find a polynomial function $f$ that has the given zeros satisfying the given condition.

**a.** $2, -3, 5$ and $f(4) = 10$

**b.** $0, -\frac{2}{3}, \frac{1}{2}, 3$ $f(1) = \frac{5}{4}$

**14.** Find all rational zeros of:

**a.** $f(x) = 6x^3 - 7x^2 - 9x - 2$

**b.** $f(x) = 12x^4 - 22x^3 - 12x^2 + 33x - 9$

**c.** $f(x) = -6x^5 + 5x^4 - 3x^3 - 21x^2 - x + 6$

**d.** $f(t) = t^3 + \frac{11}{6}t^2 - \frac{1}{2}t - \frac{1}{3}$

**15.** Sketch the graphs of

**a.** $f(x) = -2x^2 + 5x - 2$

**b.** $f(x) = x^2 + 2x + 2$

# UNIT 3

# EXPONENTIAL AND LOGARITHMIC FUNCTIONS

## Unit Outcomes

By the end of this unit, you will be able to

- Apply the laws of exponents for real exponents.
- Define exponential and logarithmic functions.
- Identify domain and range of exponential and logarithmic functions.
- Solve mathematical problems involving exponents and logarithms.

## Unit Contents


- ✓ base
- ✓ characteristics
- ✓ common logarithm
- ✓ exponent
- ✓ exponential equation
- ✓ logarithm of a number
- ✓ logarithm
- ✓ exponential function
- ✓ logarithmic equation
- ✓ logarithmic function
- ✓ mantissa
- ✓ logarithmic equation
- ✓ natural logarithm
- ✓ power
- ✓ Asymptote
- ✓ antilogarithm

# 3.1 Exponents and Logarithms

## Introduction

Two of the most important functions that occur in mathematics and its applications are the exponential function $f(x) = a^x$ and its inverse function, the logarithmic function $g(x) = \log_a x$. Such functions arise in many applications and are powerful mathematical tools for solving real life problems such as analyzing population growth, decay of radioactive substances, calculating compound interest in accounting, etc.

In this unit, we will investigate their various properties and learn how they can be used in solving real life problems.

## 3.1.1 Exponents

Whenever we use expressions like $7^3$ or $2^5$, we are using exponents. The symbol $2^5$ means $\underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5\text{ factors}}$. This symbol is read as 'the $5^{\text{th}}$ power of 2' or '2 raised to 5'. The expression $2^5$ is just a shorthand way of writing the product of five twos. The number 2 is called the base, and 5 the exponent.

Similarly, if $a$ is any real number, then $a^4$ stands for $a \times a \times a \times a$. Here $a$ is the base, and 4 is the exponent.

## Activity 3.1

1. Identify the base and the exponent of each of the following.

   a) $3^4$ b) $(-3)^4$ c) $\left(\frac{3}{5}\right)^5$ d) $(-1)^9$

2. Find the values of each of the following.

   a) $(-1)^1$ b) $(-1)^4$ c) $\left(\frac{3}{5}\right)^1$ d) $(-2)^7$

   e) $-2^4$ f) $(-2)^4$ g) $\left(-\frac{2}{3}\right)^4$

## Definition 3.1

For a natural number $n$ and a real number $a$, the symbol $a^n$, read as "the $n^{th}$ power of $a$" or "$a$ raised to $n$", is defined as follows:

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}.$$

In $a^n$, $a$ is called the base and $n$ is called the exponent.

Special names are used when the exponent is 2 or 3. The expression $b^2$ is usually read as '$b$ squared', and the expression $b^3$ as '$b$ cubed'. Thus, 'two cubed' means $2^3 = 2 \times 2 \times 2 = 8$.

Note that, in $(-a)^n$ the base is $-a$ but in $-a^n$ the base is only $a$.

For example,

$(-3)^2 = (-3) \times (-3) = 9$ but $-3^2 = -(3 \times 3) = -9$.

$(3a)^3 = 3a \times 3a \times 3a = 27a^3$ but $3a^3 = 3 \times a \times a \times a = 3a^3$.

### Example 1

Evaluate the following.

**a.** $2^3$ **b.** $-2^3$ **c.** $(-2)^3$ **d.** $-(-2)^3$ **e.** $(-4t)^3$

**Solution:**

**a.** $2^3 = 2 \times 2 \times 2 = 8$

**b.** $-2^3 = -(2 \times 2 \times 2) = -8$

**c.** $(-2)^3 = (-2) \times (-2) \times (-2) = -8$

**d.** $-(-2)^3 = -[(-2) \times (-2) \times (-2)] = -(-8) = 8$

**e.** $(-4t)^3 = -4t \times -4t \times -4t = -64t^3$

## Laws of Exponents

If $a$ is any real number and $n$ is a positive integer then $a^n$ means $\underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}}$. The laws for the behaviors of exponents follow naturally from this meaning of $a^n$ for a natural number $n$.

If $a$ is a real number and $m$ and $n$ are natural numbers, then

$$a^m \times a^n = \underbrace{a \times a \times \ldots \times a}_{m \text{ factors}} \times \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$$

$$= \underbrace{a \times a \times a \times \ldots \times a \times a \times a \times \ldots \times a}_{m+n \text{ factors}}$$

$$= a^{m+n}.$$

**Law 1**. $a^m \times a^n = a^{m+n}$, where $a$ is a real number and $m$ and $n$ are natural numbers.

That is, to multiply two numbers in exponential form (with the same base), we add their exponents.

### Example 2

$$2^3 \times 2^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$$

$$= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{7 \text{ factors}}$$

$$= 2^7$$

$$= 2^{3+4}.$$

If $a$ is a real number different from zero and $m$ and $n$ are natural numbers with $n > m$, then

$$\frac{a^n}{a^m} = \frac{\underbrace{a \times a \times a \times a \times \ldots \times a}_{n \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}} = \frac{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}} \times \underbrace{a \times a \times \ldots \times a}_{n-m \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}}$$

$$= \frac{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a}_{m \text{ factors}}} \times \underbrace{a \times a \times \ldots \times a}_{n-m \text{ factors}}$$

$$= \underbrace{a \times a \times \ldots \times a}_{n-m \text{ factors}} = a^{n-m}.$$

**Law 2.** $\frac{a^n}{a^m} = a^{n-m}$, where $a$ is a real number different from zero and $m$ and $n$ are natural numbers.

Since we have not yet given any meaning to zero and negative exponents, $n$ must be greater than $m$ for law 2 to make sense.

## Example 3

$$\frac{3^6}{3^3} = \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3}$$

$$= 3 \times 3 \times 3 \times \frac{3\times3\times3}{3\times3\times3}$$

$$= 3 \times 3 \times 3$$

$$= 3^3 = 3^{6-3}$$

**Law 3.** $(a^m)^n = a^{m\times n}$, where $a$ is a real number and $m$ and $n$ are natural numbers.

## Example 4

$$(3^2)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3)$$

$$= 3^6 = 3^{2\times3}.$$

From the definition of exponents, we know that if $n$ is a natural number, then

$$(ab)^n = \underbrace{(ab) \times (ab) \times . . . \times (ab)}_{n \text{ factors}}$$

$$= \underbrace{a \times a \times . . . \times a}_{n \text{ factors}} \times \underbrace{b \times b \times . . . \times b}_{n \text{ factors}} = a^n b^n$$

**Law 4.** $(ab)^n = a^n b^n$, where $a$ and $b$ are real numbers and $n$ is a natural number.

### Example 5

$$(2 \times 3)^3 = (2 \times 3) \times (2 \times 3) \times (2 \times 3)$$
$$= (2 \times 2 \times 2) \times (3 \times 3 \times 3)$$
$$= 2^3 \times 3^3$$

If $a$ and $b$ are real numbers, $b \neq 0$ and $n$ is a natural number, then by the definition of exponent,

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}_{n \text{ factors}} = \frac{\overbrace{a \times a \times \ldots \times a}^{n \text{ factors}}}{\underbrace{b \times b \times \ldots \times b}_{n \text{ factors}}}$$
$$= \frac{a^n}{b^n}$$

**Law 5.** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $a$ and $b$ are real numbers, $b \neq 0$ and $n$ is a natural number.

### Example 6

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3} = \frac{2^4}{3^4}$$

## Exercise 3.1

1. Evaluate the following.

   a) $4^2$ b) $-4^2$

   c) $(-4)^2$ d) $-(-3)^3$

2. Simplify the exponential expressions.

   a) $a^3 \times a^2$ b) $\frac{a^5}{a^2}$ c) $(a^2)^3$

   d) $(2a)^4$ e) $\left(\frac{a}{2}\right)^2$

## Zero and Negative Exponents

**Activity 3.2**

Evaluate each of the following using the law $\frac{a^m}{a^n} = a^{m-n}$.

a) $\frac{3^2}{3^2}$; what is the value of $3^0$ ?

b) $\frac{(-3)^2}{(-3)^2}$; what is the value of $(-3)^0$ ?

c) $\frac{(0.1)^2}{(0.1)^2}$; what is the value of $(0.1)^0$ ?

Let us begin by extending $a^n$ to include an exponent equal to 0 $(n = 0)$. We want to make sense of the expression $a^0$ in such a way that Laws 1, 2 and 3 hold. What happens to law 2 when $m = n$? Law 2 gives,

$$\frac{a^n}{a^n} = a^{n-n}$$

$$1 = a^0$$

It doesn't make sense to talk about a number being multiplied by itself 0 times. However, if we want law 2 to continue to be valid when $n = m$ then we must define the expression $a^0$ to mean the number 1.

If $a \neq 0$ then we define $a^0$ to be equal to 1. We do not attempt to give any meaning to the expression $0^0$. It remains undefined.

Using this definition, we can check laws 1 and 2 also remain valid.

That is, $a^m \times a^0 = a^m \times 1 = a^m = a^{m+0}$ and $\frac{a^n}{a^0} = \frac{a^n}{1} = a^n = a^{n-0}$.

To come up with a suitable meaning for negative exponents, we can take $n < m$ in law 2. For example, let us try $n = 2$ and $m = 3$.

$\frac{a^2}{a^3} = a^{2-3} = a^{-1}$.

But, $\frac{a^2}{a^3} = \frac{a \times a}{a \times a \times a} = \frac{1}{a}$. Therefore, $\frac{1}{a} = a^{-1}$.

Similarly, for $a \neq 0$, $\frac{a^5}{a^8} = a^{-3}$ implies $\frac{1}{a^3} = a^{-3}$ and $\frac{a^8}{a^{20}} = a^{-12}$ implies $\frac{1}{a^{12}} = a^{-12}$.

## Definition 3.2 Zero and Negative Exponents

If $n$ is a positive integer and $a \neq 0$, then

1) $a^0 = 1$ and $0^0$ is undefined.

2) $a^{-n} = \frac{1}{a^n}$.

### Example 7

Evaluate the following.

**a)** $1^0$ **b)** $(-10)^0$

**c)** $\left(\frac{1}{2}\right)^0$ **d)** $(0.123)^0$

**Solution:**

**a)** $1^0 = 1$ **b)** $(-10)^0 = 1$

**c)** $\left(\frac{1}{2}\right)^0 = 1$ **d)** $(0.123)^0 = 1$

### Example 8

Evaluate the following.

**a)** $3^{-2}$ **b)** $\left(\frac{2}{3}\right)^{-3}$ **c)** $\left(\frac{1}{2}\right)^{-2}$

**Solution:**

**a)** $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

**b)** $\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{2^3}{3^3}} = \frac{3^3}{2^3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

**c)** $\left(\frac{1}{2}\right)^{-2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1^2}{2^2}} = \frac{2^2}{1^2} = 2^2 = 4$

## Exercise 3.2

Evaluate the following.

a) $4^0$ b) $(-11)^0$ c) $\left(\frac{22}{55}\right)^0$ d) $2^{-3}$

e) $10^{-2}$ f) $\left(\frac{3}{4}\right)^{-3}$ g) $\left(\frac{1}{2}\right)^{-5}$

## Laws for Integer Exponents:

**For real numbers *a* and *b* and integers *m* and *n*, the following laws of exponents hold true.**

1. $a^m \times a^n = a^{m+n}$ . . . law of multiplication of powers of the same base.
2. $\frac{a^m}{a^n} = a^{m-n}$ . . . law of division of powers of the same base.
3. $(a^m)^n = a^{mn}$ . . . law of power of a power.
4. $(a \times b)^n = a^n \times b^n$ . . . law of a power of a product.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ . . . law of a power of a quotient.

### Example 9

Simplify each of the following.

**a)** $a^{-2} \times a^5$ **b)** $(3a)^4 \times (3a)^{-2}$ **c)** $\frac{5^4}{5^6}$ **d)** $(a^2)^{-3}$

**e)** $x^2 \times x^{-3} \times x^4$ **f)** $\left(\frac{3a}{16}\right)^2$ **g)** $(81)^t \times 9^{2t}$ **h)** $(-5x \times 3y)^2$

**Solution:**

**a)** $a^{-2} \times a^5 = a^{-2+5} = a^3$

**b)** $(3a)^4 \times (3a)^{-2} = (3a)^{4+(-2)} = (3a)^2 = 3^2a^2 = 9a^2$

**c)** $\frac{5^4}{5^6} = 5^{4-6} = 5^{-2} = \frac{1}{25}$

**d)** $(a^2)^{-3} = a^{2\times(-3)} = a^{-6} = \frac{1}{a^6}$

**e)** $x^2 \times x^{-3} \times x^4 = (x^2 \times x^{-3}) \times x^4 = x^{-1} \times x^4 = x^{-1+4} = x^3$

**f)** $\left(\frac{3a}{16}\right)^2 = \frac{3^2 \times a^2}{(16)^2} = \frac{9a^2}{256}$

**g)** $(81)^t \times 9^{2t} = 3^{4t} \times (3^2)^{2t} = 3^{4t} \times 3^{4t} = 3^{4t+4t} = 3^{8t}$

**h)** $(-5x \times 3y)^2 = (-5x)^2 \times (3y)^2$

$$= (-5)^2x^2 \times 3^2y^2 = 25x^2 \times 9y^2 = 225x^2y^2$$

## Exercise 3.3

Simplify the exponential expressions using laws of exponents.

a) $x^{-3} \times x^4$ b) $\frac{2^5}{2^7}$ c) $\frac{3^{-5}}{3^{-3}}$

d) $(-2x \times 4y)^2$ e) $x^{-3} \times x^{-2}$ f) $(4y)^2 \times (8y)^{-3}$

g) $2^t \times 2^{3t} \times 2^{2t}$ h) $\frac{(2x)^2}{(2x)^4}$ i) $\frac{(-3x)^2}{(-3x)^4}$

j) $(3^2)^{2n}$ k) $(a^y)^{-1}$ l) $(a^{3x})^4$

m) $(2a^{-3} \times b^2)^{-2}$ n) $\frac{(a^2)^{-3} \times (a^3)^4}{a^{10}}$ o) $\left(\frac{m^{-5}n^2}{n^{-2}m^6}\right)^{-2}$

## The Rational Exponent

Extend the definition of exponents even further to include rational numbers. For example, to define powers like $a^{\frac{1}{2}}$, consider $9^{\frac{1}{2}}$.

Applying law 3 and taking the square of $9^{\frac{1}{2}}$, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{\left(\frac{1}{2}\right)\cdot(2)} = 9^1 = 9$. Thus, $9^{\frac{1}{2}}$ is a number that yields 9 when squared. There are two numbers whose square is 9. They are 3 and −3. We define $9^{\frac{1}{2}}$ to be the positive square root of 9. That is, 3. To avoid ambiguity, we define $a^{\frac{1}{2}}$ as the non-negative number that yields $a$ when squared. Thus, $9^{\frac{1}{2}} = 3$.

In general, $a^{\frac{1}{2}}$ is defined to be the positive square root of $a$, which can also be written as $\sqrt{a}$ . So $a^{\frac{1}{2}} = \sqrt{a}$.

Of course, $a$ must be positive if $a^{\frac{1}{2}}$ is meaningful because if we take any real number and multiply it by itself, then we get a positive number.

We can arrive at the definition of $a^{\frac{1}{3}}$ in the same way as we did for $a^{\frac{1}{2}}$. For example, if we cube $8^{\frac{1}{3}}$, we get $\left(8^{\frac{1}{3}}\right)^3 = (8)^{\left(\frac{1}{3}\times 3\right)} = 8$. Thus, $8^{\frac{1}{3}}$ is the number that yields 8 when cubed. Since $2^3 = 8$ we have $8^{\frac{1}{3}} = 2$. Similarly, $(-27)^{\frac{1}{3}} = -3$.

This time we have no trouble giving a meaning to $(-27)^{\frac{1}{3}}$ even though $-27 < 0$. There is a number when multiplied by itself 3 times gives -27, namely $-3$, so

$(-27)^{\frac{1}{3}} = -3$.

Thus, we define $a^{\frac{1}{3}}$ (called the cube root of $a$) as the quantity that yields $a$ when cubed.

**Definition 3.3 The Rational Exponent $a^{\frac{1}{n}}$**

If $a$ is positive, then $a^{\frac{1}{n}}$ is defined to be a positive number whose $n^{th}$ power is equal to $a$. This number is called the $n^{th}$ root of $a$ and sometimes written as $\sqrt[n]{a}$.

If $n$ is even and $a$ is negative, $a^{\frac{1}{n}}$ cannot be defined because raising any number to an even power result in a positive number.

If $n$ is odd and $a$ is negative, $a^{\frac{1}{n}}$ can be defined. It is a negative number whose $n^{th}$ power is equal to $a$.

## Example 10

Express in the form $a^{\frac{1}{n}}$ and evaluate the following.

**a)** $\sqrt[4]{16}$ **b)** $\sqrt[3]{8}$ **c)** $\sqrt[3]{-8}$ **d)** $\sqrt[4]{-16}$

**Solution:**

**a)** Since $2^4 = 16$, $\sqrt[4]{16} = (16)^{\frac{1}{4}} = 2$

**b)** Since $2^3 = 8$, $\sqrt[3]{8} = (8)^{\frac{1}{3}} = 2$

**c)** Since $(-2)^3 = -8$, $\sqrt[3]{-8} = (-8)^{\frac{1}{3}} = -2$

**d)** $\sqrt[4]{-16} = (-16)^{\frac{1}{4}}$ is not a real number because there is no real number $a$ such that $a^4$ is $-16$

## Exercise 3.4

Express in the form $a^{\frac{1}{n}}$ and evaluate the following

a) $\sqrt[4]{81}$ b) $\sqrt[5]{32}$ c) $\sqrt[3]{125}$

d) $\sqrt[3]{-27}$ e) $-\sqrt[3]{-1000}$ f) $\sqrt[4]{-10000}$

## Activity 3.3

Simplify the following expressions.

a) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ b) $\sqrt{2} \times \sqrt{2}$ c) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ d) $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$

So far, we have defined $a^{\frac{1}{n}}$, where $n$ is a natural number. With the help of the third law for exponent, we can notice that,

$$\frac{m}{n} = m \times \frac{1}{n}.$$

So, if law 3 is to hold then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

Therefore, we can define the expression $a^{\frac{m}{n}}$, where $m$ and $n$ are natural numbers and $\frac{m}{n}$ is reduced to lowest term as in definition 3.4.

## Definition 3.4 The Rational Exponent $a^{\frac{m}{n}}$

If $a^{\frac{1}{n}}$ is a real number, then $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$ (that is, the $n^{th}$ root of $a$ raised to the $m^{th}$ power).

We can also define negative rational exponents:

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} \quad (a \neq 0)$$

**Note**

The laws of exponents discussed earlier for integral exponents hold true for rational exponents.

### Example 11

Evaluate the following expressions.

**a)** $4^{\frac{1}{3}} \times 16^{\frac{1}{3}}$ **b)** $\frac{3^{\frac{1}{2}}}{27^{\frac{3}{2}}}$ **c)** $27^{\frac{2}{3}}$ **d)** $\left(3m^{\frac{1}{2}} \times 4n^{\frac{3}{2}}\right)^2$ **e)** $(-32)^{-\frac{3}{5}}$

**Solution:**

**a)** $4^{\frac{1}{3}} \times 16^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} \times (2^4)^{\frac{1}{3}} = 2^{\frac{2}{3}} \times 2^{\frac{4}{3}} = 2^{\frac{2}{3}+\frac{4}{3}} = 2^2 = 4$

**b)** $\frac{3^{\frac{1}{2}}}{27^{\frac{3}{2}}} = \frac{3^{\frac{1}{2}}}{3^{\frac{9}{2}}} = 3^{\frac{1}{2}-\frac{9}{2}} = 3^{-\frac{8}{2}} = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

**c)** $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = 3^2 = 9$

**d)** $\left(3m^{\frac{1}{2}} \times 4n^{\frac{3}{2}}\right)^2 = \left(3m^{\frac{1}{2}}\right)^2 \times \left(4n^{\frac{3}{2}}\right)^2 = 3^2m \times 4^2n^3 = 9m \times 16n^3 = 144mn^3$

**e)** $(-32)^{-\frac{3}{5}} = \frac{1}{(-32)^{\frac{3}{5}}} = \frac{1}{\left((-32)^{\frac{1}{5}}\right)^3} = \frac{1}{(-2)^3} = -\frac{1}{8}$

## Exercise 3.5

Simplify each of the following expressions.

a) $2^{\frac{1}{3}} \times 4^{\frac{4}{3}}$ b) $\frac{16^{\frac{2}{5}}}{8^{\frac{6}{5}}}$ c) $100^{\frac{3}{2}}$ d) $\left(a^{\frac{1}{4}} \times 3b^{\frac{5}{2}}\right)^4$

e) $(-27)^{-\frac{2}{3}}$  f) $\left(\frac{x^{\frac{2}{3}}}{y^{-\frac{1}{2}}}\right)^{-6}$  g) $\frac{(a^2)^{-\frac{1}{4}}\times(a^3)^{\frac{2}{9}}}{a^{\frac{1}{2}}}$  h) $\left(\frac{m^{\frac{1}{8}}n^{-\frac{1}{3}}}{n^{\frac{2}{3}}m^{\frac{1}{4}}}\right)^4$

Radical notation is an alternative way of writing an expression with rational exponents.

## Definition 3.5 The $n^{th}$ root

If $a^{\frac{1}{n}}$ is a real number and $m$ an integer then

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} \quad (or\ a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m).$$

### Example 12

Express in the form $a^{\frac{m}{n}}$, with $a$ being a prime number.

**a)** $\sqrt[3]{4}$  **b)** $\sqrt[5]{27}$  **c)** $\frac{\sqrt[3]{8}}{\sqrt[3]{4}}$  **d)** $(\sqrt[3]{625})^2$  **e)** $\sqrt[3]{\sqrt[2]{64}}$

**Solution:**

**a)** $\sqrt[3]{4} = 4^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} = 2^{\frac{2}{3}}$.

**b)** $\sqrt[5]{27} = (27)^{\frac{1}{5}} = (3^3)^{\frac{1}{5}} = 3^{\frac{3}{5}}$

**c)** $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \frac{(2^3)^{\frac{1}{3}}}{(2^2),^{\frac{1}{3}}} = \frac{2^1}{2^{\frac{2}{3}}} = 2^{1-\frac{2}{3}} = 2^{\frac{1}{3}}$

**d)** $(\sqrt[3]{625})^2 = \left((5^4)^{\frac{1}{3}}\right)^2 = \left(5^{\frac{4}{3}}\right)^2 = 5^{\frac{8}{3}}$

**e)** $\sqrt[3]{\sqrt[2]{64}} = \sqrt[3]{(64)^{\frac{1}{2}}} = \sqrt[3]{(8^2)^{\frac{1}{2}}} = \sqrt[3]{8} = \sqrt[3]{2^3} = (2^3)^{\frac{1}{3}} = 2$

## Exercise 3.6

Express in the form $a^{\frac{m}{n}}$, with $a$ being a prime number.

a) $\sqrt[3]{81}$  b) $\sqrt[4]{32}$  c) $\frac{\sqrt[3]{25}}{\sqrt[3]{5}}$  d) $\frac{\sqrt[5]{40}}{\sqrt[5]{5}}$

e) $(\sqrt[4]{27})^6$ f) $(\sqrt[3]{121})^2$ g) $\sqrt[3]{\sqrt[2]{\frac{1}{1000000}}}$

## Irrational Exponents

The expressions $3^{\sqrt{2}}$, $2^{2\sqrt{3}}$, $5^{\pi}$ are powers with irrational exponents.

What is the value of $3^{\sqrt{2}}$ ?

Using calculator, the value of $\sqrt{2} = 1.41421356...$

Therefore, $3^{\sqrt{2}} = 3^{1.41421356...}$

It is not possible to calculate $3^{\sqrt{2}} = 3^{1.41421356...}$ because $1.41421356...$ has infinite decimals. But one can approximate the value of $3^{1.41421356...}$ as follows.

To approximate $3^{1.41421356...}$

$3^1 = 3$

$3^{1.4} = 3^{\frac{14}{10}} = 4.65553672$, to eight decimal places

( $3^{\frac{14}{10}}$ is an expression with rational exponent, use a calculator to find its value)

$3^{1.41} = 3^{\frac{141}{100}} = 4.70696500$

$3^{1.414} = 3^{\frac{1414}{1000}} = 4.72769503$

$3^{1.4142} = 4.72873393$

$3^{1.41421} = 4.72878588$

$3^{1.4142135} = 4.72880406$

$3^{1.41421356} = 4.72880437$

As we can see from the above list, the values of

$3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, 3^{1.41421}, 3^{1.4142135}, 3^{1.41421356}$

are approaching to some number. For example, the first six decimals of the values of $3^{1.4142135}$ and $3^{1.41421356}$ are the same. That is 4.728804.

By continuing the approximation of $3^{\sqrt{2}}$ like above we can say $3^{\sqrt{2}} \cong 4.72880437$ to eight decimal places.

Now, if we define $3^x$ then $3^{\sqrt{2}}$ is the real number that $3^x$ approaches when $x$ gets closer and closer to $\sqrt{2}$.

In general, if we define $a^x$ and $b$ is an irrational number, then $a^b$ is a real number that $a^x$ approaches when $x$ gets closer and closer to $b$.

The above statement about irrational exponents suggests that the expression $a^x$ is defined not only for integral and rational exponents but also for irrational exponents.

The laws of exponents discussed earlier for integral and rational exponents continue to hold true for irrational exponents.

## Example 13

Simplify each of the following.

**a)** $3^{\sqrt{2}} \times 3^{\sqrt{2}}$ **b)** $\left(4^{\sqrt{2}}\right)^3$ **c)** $\left(3^{\sqrt{2}}\right)^{\sqrt{2}}$

**d)** $\frac{2^{\sqrt{3}+2}}{2^{\sqrt{3}-3}}$ **e)** $5^{\sqrt{2}} \times 5^{\sqrt{3}}$ **f)** $\left(5^{\sqrt{2}}\right)^{\sqrt{3}}$

**Solution:**

**a)** $3^{\sqrt{2}} \times 3^{\sqrt{2}} = 3^{\sqrt{2}+\sqrt{2}} = 3^{2\sqrt{2}} = (3^2)^{\sqrt{2}} = 9^{\sqrt{2}}$

**b)** $\left(4^{\sqrt{2}}\right)^3 = 4^{3\times\sqrt{2}} = (4^3)^{\sqrt{2}} = 64^{\sqrt{2}}$

**c)** $\left(3^{\sqrt{2}}\right)^{\sqrt{2}} = 3^{\sqrt{2}\times\sqrt{2}} = 3^2 = 9$

**d)** $\frac{2^{\sqrt{3}+2}}{2^{\sqrt{3}-3}} = \frac{2^{\sqrt{3}}\times 2^2}{2^{\sqrt{3}}\times 2^{-3}} = \frac{2^2}{2^{-3}} = 2^{2-(-3)} = 2^5 = 32$

or $\frac{2^{\sqrt{3}+2}}{2^{\sqrt{3}-3}} = (2)^{(\sqrt{3}+2)-(\sqrt{3}-3)} = 2^5 = 32$

**e)** $5^{\sqrt{2}} \times 5^{\sqrt{3}} = 5^{\sqrt{2}+\sqrt{3}}$

**f)** $\left(5^{\sqrt{2}}\right)^{\sqrt{3}} = 5^{\sqrt{2}\times\sqrt{3}} = 5^{\sqrt{6}}$

## Exercise 3.7

Simplify each of the following.

a) $2^{\sqrt{3}} \times 2^{\sqrt{3}}$ b) $\left(5^{\sqrt{2}}\right)^2$ c) $\left(\sqrt[3]{8}\right)^{-2}$ d) $\sqrt{3}^{\sqrt{2}} \times \sqrt{3}^{\sqrt{8}}$

e) $\frac{3^{\sqrt{2}+3}}{3^{\sqrt{2}-1}}$ f) $\left(2^{\sqrt{3}}\right)^{\sqrt{27}}$ g) $\frac{3^{\sqrt{2}} \times 9^{\sqrt{8}}}{27^{\sqrt{18}}}$ h) $\frac{(5^{\sqrt{3}})^2 \times 5^{-\sqrt{12}} \times 25^{\sqrt{3}}}{5^{\sqrt{27}}}$

## 3.1.2 Logarithms

In the exponential equation $2^3 = 8$, the base is 2 and the exponent is 3. We write this equation in logarithm form as $\log_2 8 = 3$. We read this as "the logarithm of 8 to the base 2 is 3".

The logarithm to base $a$ of a number $x > 0$ (written $\log_a x$) is that power to which $a$ must be raised to obtain the number $x$.

For example,

$\log_3 9 = 2$ because $3^2 = 9$.

$\log_3 \left(\frac{1}{9}\right) = -2$ because $3^{-2} = \frac{1}{9}$.

$\log_5 1 = 0$ because $5^0 = 1$.

### Definition 3.6

For a $> 0$, $a \neq 1$, and $c > 0$

$$\log_a c = b \text{ if and only if } a^b = c.$$

### Example 1

Convert each of the following to logarithmic statement.

**a)** $2^5 = 32$ **b)** $3^4 = 81$ **c)** $4^{\frac{1}{2}} = 2$

**Solution:**

**a)** From $2^5 = 32$, we have $\log_2 32 = 5$.

**b)** From $3^4 = 81$, we have $\log_3 81 = 4$.

**c)** From $4^{\frac{1}{2}} = 2$, we have $\log_4 2 = \frac{1}{2}$.

### Example 2

Convert each of the following to exponential statement.

**a)** $\log_2 128 = 7$  **b)** $\log_3\left(\frac{1}{27}\right) = -3$  **c)** $\log_{10}\sqrt[3]{10} = \frac{1}{3}$

**Solution:**

**a)** From $\log_2 128 = 7$, we have $2^7 = 128$.

**b)** From $\log_3\left(\frac{1}{27}\right) = -3$, we have $3^{-3} = \frac{1}{27}$.

**c)** From $\log_{10}\sqrt[3]{10} = \frac{1}{3}$, we have $10^{\frac{1}{3}} = \sqrt[3]{10}$.

### Example 3

Find the value of each of the following logarithms.

**a)** $\log_2 64$  **b)** $\log_3\left(\frac{1}{81}\right)$  **c)** $\log_{10} 100$  **d)** $\log_{10} 0.01$

**Solution:**

**a)** Since $64 = 2^6$ , the exponent to which we raise 2 to get 64 is 6.

So, $\log_2 64 = 6$.

**b)** Since $\frac{1}{81} = \frac{1}{3^4} = 3^{-4}$, the exponent to which we raise 3 to get $\frac{1}{81}$ is $-4$.

Hence, $\log_3\left(\frac{1}{81}\right) = -4$ .

**c)** The exponent to which we raise 10 to get 100 is 2 as $10^2 = 100$. Therefore, $\log_{10} 100 = 2$.

**d)** Since $0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$, the exponent to which we raise 10 to get $\frac{1}{100}$ is $-2$. Thus, $\log_{10} 0.01 = -2$ .

## Exercise 3.8

1. Write the equivalent logarithmic statement for the following equations.

   a) $2^{10} = 1{,}024$  b) $2^{-6} = \frac{1}{64}$  c) $\sqrt[3]{125} = 5$  d) $27^{-\frac{2}{3}} = \frac{1}{9}$

2. Write the equivalent exponential statement for the following equations.

a) $\log_{10} 1000 = 3$ b) $\log_8 \sqrt{64} = 1$

c) $\log_{10} 0.001 = -3$ d) $\log_3 \frac{1}{27} = -3$

3. Find the values of following logarithms.

a) $\log_3 27$ b) $\log_4 16$

c) $\log_{100} 0.001$ d) $\log_{\sqrt{49}} 7$

## Properties of logarithms

The following properties follow directly from the definition of the logarithm with base $a > 0$ and $a \neq 1$

| Properties of Logarithms |
|---|
| **1.** $\log_a 1 = 0$ because $a^0 = 1$. |
| **2.** $\log_a a = 1$ because $a^1 = a$. |
| **3.** $\log_a a^p = p$ and $a^{\log_a p} = p$ . . . Inverse property. |
| **4.** If $\log_a \mathrm{M} = \log_a \mathrm{N}$, then $\mathrm{M} = \mathrm{N}$ . . . One-to-One property. |

### Example 4

Use properties of logarithms to answer the following questions.

**a)** Find $p$ such that $\log_2 p = \log_2 5$. **b)** Simplify $\log_2 2^p$.

**c)** Find $p$ such that $\log_3 3 = p$. **d)** Simplify $5^{\log_5 p}$.

**Solution:**

**a)** Using property 4, we can see that $p = 5$.

**b)** Using property 3, it follows that $\log_2 2^p = p$.

**c)** Using property 2, we can conclude that $p = 1$.

**d)** Using property 3, it follows that $5^{\log_5 p} = p$.

## Exercise 3.9

Using properties of logarithms give answer for the following questions.

**a)** Find $p$ such that $\log_3 p = \log_3 4$. **b)** Simplify $\log_5 25^p$.

**c)** Find $p$ such that $\log_6 6 = p$. **d)** Simplify $2^{\log_2 3p}$.

## Laws of logarithms

We now establish laws of logarithms. The laws are represented by theorems and we prove the theorems based on the corresponding laws of exponents.

### Theorem 3.1 Logarithms of products

For any positive numbers $M$, $N$ and $a > 0$ and $a \neq 1$,

$$\log_a MN = \log_a M + \log_a N$$

(The logarithm of a product is the sum of the logarithms of the factors.)

**Note**

This property of logarithms corresponds to the product law for exponents:

$$a^M a^N = a^{M+N}.$$

**Proof:**

Let $\log_a M = p$ and $\log_a N = q$.

Converting to exponential equations, we get : $a^p = M$ and $a^q = N$.

Now, $MN = a^p \cdot a^q = a^{p+q}$.

This implies $MN = a^{p+q}$.

Converting back to a logarithmic equation, we obtain

$\log_a MN = p + q$.

But, $p = \log_a M$ and $q = \log_a N$.

Therefore, $\log_a MN = p + q = \log_a M + \log_a N$.

### Example 5

Express $\log_2(4 \times 8)$ as a sum of logarithms.

**Solution:**

We have $\log_2(4 \times 8) = \log_2 4 + \log_2 8$ . . . using the product law

### Example 6

Express $\log_3 5 + \log_3 8$ as a single logarithm.

**Solution:**

We have $\log_3 5 + \log_3 8 = \log_3(5 \times 8) = \log_3 40$ . . . using the product law

## Theorem 3.2 Logarithms of powers

For any positive number $M$, any real number $r$, and $a > 0$ *and* $a \neq 1$,

$$\log_a (M)^r = r \log_a M$$

(The logarithm of a power of $x$ is the exponent times the logarithms of $x$.)

**Note**

This property of logarithms corresponds to the power law for exponents:

$$(a^M)^r = a^{Mr} \cdot$$

## Proof:

Let $p = \log_a M$.

Converting to exponential equations, we get $a^p = M$.

Now, $(a^p)^r = a^{pr} = M^r$.

Converting back to a logarithmic equation, we obtain: $\log_a M^r = \log_a a^{pr} = pr$.

But, $p = \log_a M$.

Therefore, $\log_a (M)^r = r \log_a M$.

### Example 7

Use laws of logarithms to evaluate the following.

**a)** $\log_2 \sqrt{8}$ **b)** $\log_3 81$ **c)** $\log_{10} \sqrt[3]{0.01}$

**d)** $\log_4 8 + \log_4 2$ **e)** $\log_{\frac{1}{2}} \left(\frac{1}{4}\right)$

**Solution:**

**a)** $\log_2 \sqrt{8} = \log_2 8^{\frac{1}{2}} = \frac{1}{2}\log_2 2^3 = \frac{1}{2}(3 \log_2 2) = \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$.

**b)** $\log_3 81 = \log_3 3^4 = 4 \log_3 3 = 4 \times 1 = 4$.

**c)** $\log_{10} \sqrt[3]{0.01} = \log_{10}(0.01)^{\frac{1}{3}}$

$= \frac{1}{3}\log_{10} 0.01$

$= \frac{1}{3}\log_{10} \frac{1}{100}$

$= \frac{1}{3}\log_{10} \frac{1}{10^{-2}}$

$= \frac{1}{3}\log_{10} 10^{-2} = \frac{1}{3}(-2\log_{10} 10) = \frac{1}{3} \times (-2) \times 1 = -\frac{2}{3}.$

**d)** $\log_4 8 + \log_4 2 = \log_4(8 \times 2) = \log_4 16 = \log_4 4^2 = 2\log_4 4 = 2 \times 1 = 2$

**e)** $\log_{\frac{1}{2}}\left(\frac{1}{4}\right) = \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^2 = 2\log_{\frac{1}{2}}\left(\frac{1}{2}\right) = 2 \times 1 = 2$

## Exercise 3.10

Use laws of logarithm to find the values of

a) $\log_3 \sqrt{3}$ b) $\log_6 36$ c) $\log_2\left(\frac{1}{4}\right)$

d) $\log_{\left(\frac{1}{3}\right)}\left(\frac{1}{81}\right)$ e) $\log_{10} \sqrt[3]{\frac{1}{1000}}$ f) $\log_8 32 + \log_8 2$

g) $\log_2 6 + \log_2\left(\frac{1}{12}\right)$ h) $\log_3 10 + \log_3\left(\frac{6}{5}\right) + \log_3\left(\frac{9}{4}\right)$ i) $\frac{1}{2}\log_4 8 + \log_4 \sqrt{2}$

### Theorem 3.3 Logarithms of quotients

For any positive numbers $M$, $N$, and $a > 0$ and $a \neq 1$,

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N.$$

(The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.)

### Proof:

The proof follows from the product law and the power law.

$$\log_a\left(\frac{M}{N}\right) = \log_a MN^{-1}$$

$= \log_a M + \log_a N^{-1}$ . . . using product law

$= \log_a M + (-1) \log_a N$ . . . using power law

$= \log_a M - \log_a N$.

### Example 8

Use laws of logarithms to evaluate the following.

a) $\log_3 54 - \log_3 2$

b) $\log_{10} \sqrt{2000} - \log_{10} \sqrt{20}$

c) $\log_5 30 + \log_5 15 - \log_5 2$

**Solution:**

a) $\log_3 54 - \log_3 2 = \log_3 \frac{54}{2} = \log_3 27 = \log_3 3^3 = 3 \log_3 3 = 3 \times 1 = 3$

b) $\log_{10} \sqrt{2000} - \log_{10} \sqrt{20} = \log_{10} \left(\frac{\sqrt{2000}}{\sqrt{20}}\right)$

$= \log_{10} \sqrt{\frac{2000}{20}} = \log_{10} \sqrt{100} = \log_{10} 10 = 1$

c) $\log_5 30 + \log_5 15 - \log_5 2 = (\log_5 30 + \log_5 15) - \log_5 2$

$= \log_5 (30 \times 15) - \log_5 2$

$= \log_5 450 - \log_5 2$

$= \log_5 \left(\frac{450}{2}\right) = \log_5 225 = \log_5 5^3 = 3 \log_5 5 = 3 \times 1 = 3$

### Exercise 3.11

Use laws of logarithms to find the values of

a) $\log_5 50 - \log_5 2$

b) $\log_3 4 - \log_3 108$

c) $\log_{10} \sqrt{2000} - \log_{10} \sqrt{2}$

d) $\log_5 2 + \log_5 50 - \log_5 4$

e) $\log_6 9 - \log_6 15 + \log_6 10$

f) $\log_{10} 24 - 2\log_{10} 6 + \log_{10} 15$

## Theorem 3.4 Change of base

For any positive real number M, $a > 0, b > 0, a \neq 1$ and $b \neq 1$,

$$\log_a M = \frac{\log_b M}{\log_b a}$$

## Proof:

Let $p = \log_a M$. Then

$$a^p = a^{\log_a M} = M$$

$\log_b a^p = \log_b M$ . . . taking logarithm to the base $b$ of both sides

$p \log_b a = \log_b M$ . . . using power law

$$p = \frac{\log_b M}{\log_b a}$$

Therefore, $p = \log_a M = \frac{\log_b M}{\log_b a}$.

### Example 9

Use laws of logarithms to find:

a) $\log_{\sqrt{2}} 4$ b) $\log_{0.1} 100$

**Solution:**

a) $\log_{\sqrt{2}} 4 = \frac{\log_2 4}{\log_2 \sqrt{2}}$ by using base change law of logaritms

$= \frac{\log_2 2^2}{\log_2 2^{\frac{1}{2}}}$ because $4 = 2^2$ and $\sqrt{2} = 2^{\frac{1}{2}}$

$= \frac{2\log_2 2}{\frac{1}{2}\log_2 2}$ by power law of logaritms

$= \frac{2}{\frac{1}{2}}$ because $\log_2 2 = 1$

$= 2 \times \frac{2}{1}$

$= 4$

b) $\log_{0.1} 100 = \frac{\log_{10} 100}{\log_{10} 0.1}$ by using base change law of logaritms

$= \frac{\log_{10} 10^2}{\log_{10} 10^{-1}}$ because $100 = 10^2$ and $0.1 = \frac{1}{10} = 10^{-1}$

$= \frac{2\log_{10} 10}{-1\log_{10} 10}$ by power law of logaritms

$= \frac{2}{-1}$ because $\log_{10} 10 = 1$

$= -2$

## Exercise 3.12

Use the law $\log_a x = \frac{\log_b x}{\log_b a}$ to find the value of the following expressions.

a) $\log_{\sqrt{3}} 9$ b) $\log_{\sqrt{2}} 128$ c) $\log_{\left(\frac{1}{3}\right)} 243$

d) $\log_{100} 0.1$ e) $\log_4 \left(\frac{1}{2}\right)$

**Remember that:**

1. $\log_a MN \neq (\log_a M)(\log_a N)$ . . . The logarithm of a product is not the product of the logarithms.
2. $\log_a (M + N) \neq \log_a M + \log_a N$ . . . The logarithm of a sum is not the sum of the logarithms.
3. $\log_a \left(\frac{M}{N}\right) \neq \frac{\log_a M}{\log_a N}$ . . . The logarithm of a quotient is not the quotient of the logarithm.
4. $(\log_a M)^r \neq r\log_a M$ . . . The power of a logarithm is not the exponent times the logarithm.

## Logarithms to Base 10 (Common Logarithms)

For a general number $M$, $\log_{10} M$ is equal to that power to which 10 must be raised to obtain the number $M$.

### Activity 3.4

Find the value of the following common logarithms.

a) $\log_{10} 10$ b) $\log_{10} 1000$ c) $\log_{10} 1$ d) $\log_{10} 0.1$

## Definition 3.7

The logarithm to the base 10 is called common logarithm or decadic logarithm and written as $\log_{10} M$.

A common logarithim is usually written without indicating its base. For example, $\log_{10} M$ is simply denoted by $\log M$.

### Example 1

Find the value of each of the common logarithms.

a) $\log_{10} \sqrt[3]{10}$ b) $\log 0.0001$ c) $\log 200 - \log 2$ d) $\log\left(\frac{0.1}{\sqrt{100}}\right)$

**Solution:**

a) $\log_{10} \sqrt[3]{10} = \log_{10} 10^{\left(\frac{1}{3}\right)}$

$= \frac{1}{3} \log_{10} 10 = \frac{1}{3} \times 1 = \frac{1}{3}.$

b) $\log 0.0001 = \log\left(\frac{1}{10000}\right) = \log\left(\frac{1}{10^4}\right)$

$= \log 10^{-4} = -4 \log 10$ because $\log 10 = \log_{10} 10 = 1$

$= -4.$

c) $\log 200 - \log 2 = \log\left(\frac{200}{2}\right) = \log 100 = \log 10^2 = 2 \log 10 = 2 \times 1 = 2.$

## Exercise 3.13

Find the values of the following common logarithms.

a) $\log \sqrt[3]{0.1}$ b) $\log_{10}(10\sqrt{10})$ c) $\log\left(\frac{0.01}{\sqrt{1000}}\right)$ d) $\log\left(\frac{1}{\sqrt[5]{10}}\right)$ e) $\log\left(\frac{10^m}{10^n}\right)$

Suppose $p$ can be written as $p = m \times 10^c, 1 \leq m < 10$, then the logarithm of $p$ can be read from the common logarithm table (**a table that contains the common logarithm value of a number $m$ such that $1 \leq m < 10$**) which is attached at the last page of the book.

So, $\log p = \log(m \times 10^c) = \log m + \log 10^c = \log m + c$.

That is, $\log p = \log m + c$.

The common logarithm of $m$, $\log m$ is called the mantissa (fractional part) of the common logarithm of $p$ and $c$ is called the characterstic of the logarithm.

The common logarithm of any two decimal place number between 1.00 and 9.99 can be read directly from the common logarithm table.

### Example 2

Using the table of logarithm, calculate

**a)** $\log 1.23$ **b)** $\log 3.57$

**c)** $\log 2.478$ **d)** $\log 6{,}920$

### Solution:

**a)** We read the number at the intersection of row 1.2 and column 3 to find 0.0899.

| | Logarithm table | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1.0 | 0 | 0.0043 | 0.0086 | 0.0128 | 0.017 | 0.0212 | 0.0253 | 0.0294 | 0.0334 | 0.0374 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 |
| 1.1 | 0.0414 | 0.0453 | 0.0492 | 0.0531 | 0.0569 | 0.0607 | 0.0645 | 0.0682 | 0.0719 | 0.0755 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 |
| 1.2 | 0.0792 | 0.0828 | 0.0864 | 0.0899 | 0.0934 | 0.0969 | 0.1004 | 0.1038 | 0.1072 | 0.1106 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 3 |
| 1.3 | 0.1139 | 0.1173 | 0.1206 | 0.1239 | 0.1271 | 0.1303 | 0.1335 | 0.1367 | 0.1399 | 0.143 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 1.4 | 0.1461 | 0.1492 | 0.1523 | 0.1553 | 0.1584 | 0.1614 | 0.1644 | 0.1673 | 0.1703 | 0.1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |

So, $\log 1.23 = 0.0899$.

**b)** Reading the number in row 3.5 under column 7 from the common logarithm table, gives 0.5527.

So, $\log 3.57 = 0.5527$.

**c)** 2.478 in $\log 2.478$ has three numbers after decimal.

| | Logarithm table | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2.1 | 0.3222 | 0.3243 | 0.3263 | 0.3284 | 0.3304 | 0.3324 | 0.3345 | 0.3365 | 0.3385 | 0.3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 2.2 | 0.3424 | 0.3444 | 0.3464 | 0.3483 | 0.3502 | 0.3522 | 0.3541 | 0.356 | 0.3579 | 0.3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 2.3 | 0.3617 | 0.3636 | 0.3655 | 0.3674 | 0.3692 | 0.3711 | 0.3729 | 0.3747 | 0.3766 | 0.3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 2.4 | 0.3802 | 0.382 | 0.3838 | 0.3856 | 0.3874 | 0.3892 | 0.3909 | 0.3927 | 0.3945 | 0.3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 2.5 | 0.3979 | 0.3997 | 0.4014 | 0.4031 | 0.4048 | 0.4065 | 0.4082 | 0.4099 | 0.4116 | 0.4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |

To read the value of $\log 2.478$ from the logarithm table, the steps are as follow:

**Step 1.** Separate the number 2.478 in $\log 2.478$ as 2.4, 7 and 8.

**Step 2**. Read the number at the intersection of row 2.4 and column 7, this gives 0.3927.

**Step 3.** From the mean difference part of the common logarithm table, read the number at the intersection of row 2.4 and column 8, this gives 14. We write this as 0.0014.

**Step 4**. Add the values obtained in steps 2 and 3 to get $0.3927 + 0.0014 = 0.3941$. Therefore, $\log 2.478 = 0.3941$.

**d)** We have, $6{,}920 = 6.92 \times 10^3$.

So, $\log 6{,}920 = \log(6.92 \times 10^3) = \log 6.92 + \log 10^3 = \log 6.92 + 3$.

But, $\log 6.92 = 0.8401$. (reading from the table of common logarithm at the intesection of 6.9 row and column 2).

Therefore, $\log 6{,}920 = \log 6.92 + 3 = 3.8401$.

## Example 3

Identify the mantissa and characterstic of each of the common logarithms.

**a)** $\log 0.00123$ **b)** $\log 345$ **c)** $\log 0.01$

### Solution:

**a)** $0.00123 = 1.23 \times 10^{-3}$ and

$\log 0.00123 = \log(1.23 \times 10^{-3}) = \log 1.23 + \log 10^{-3} = \log 1.23 + (-3)$.

The mantissa is $\log 1.23 = 0.0899$ (Refer to the common logarithm table to find $\log 1.23$).

The characterstic is $-3$.

**b)** $345 = 3.45 \times 10^2$.

So, the mantissa is $\log 3.45 = 0.5378$ and the characterstic is 2.

**c)** We have $0.01 = 1.00 \times 10^{-2}$.

Therefore, the mantisa is $\log 1.00 = 0$ and the characterstic is -2.

## Exercise 3.14

1. Using the table of logarithm find the values of the following.

a) $\log 2.13$ b) $\log 2.99$ c) $\log 6.3$ d) $\log 6.345$

e) $\log 0.28$ f) $\log 9.99$ g) $\log 0.00008$ h) $\log 400$

2. Identify the characterstic and mantissa of the logarithm of each of the following.

a) 0.00503 b) 0.25 c) 302 d) $\frac{1}{8}$

e) 4.4 f) 9 g) 3280 h) 53.814

## Antilogarithms

Suppose $\log M = 0.8175$ . What is the value of $M$?

If $\log M = N$, then $M$ is the antilogarithm (antilog) of $N$ and write $\text{antilog}(N) = M$. When you are asked to find the antilog of a given number N you will try to find a number M such that $\log M = N$.

We can use the antilogarithm table attached at the end of the text book to read the antilog of a number.

In general, $\text{antilog}(\log M) = M$

For example, given $\log M = 0.8175$ to read the value of $M = \text{antilog}(0.8175)$ from the antilogarithm table, we follow the following steps

**Step 1**. Separate the number 0.8175 as 0.81, 7 and 5

| Anti Logarithm table | | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.79 | 6.166 | 6.18 | 6.194 | 6.209 | 6.223 | 6.237 | 6.252 | 6.266 | 6.281 | 6.295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 0.80 | 6.31 | 6.324 | 6.339 | 6.353 | 6.368 | 6.383 | 6.397 | 6.412 | 6.427 | 6.442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 0.81 | 6.457 | 6.471 | 6.486 | 6.501 | 6.516 | 6.531 | 6.546 | 6.561 | 6.577 | 6.592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 0.82 | 6.607 | 6.622 | 6.637 | 6.653 | 6.668 | 6.683 | 6.699 | 6.714 | 6.73 | 6.745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 0.83 | 6.761 | 6.776 | 6.792 | 6.808 | 6.823 | 6.839 | 6.855 | 6.871 | 6.887 | 6.902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |

**Step 2**. From the antilogarithm part read the number at the intersection of row 0.81 and column 7, this gives 6.561

**Step 3**. From the mean difference part, read the number at the intersection of row

0.81 and column 5, this gives 8. We write this as 0.008.

**Step 4**. Add the values obtained in steps 2 and 3 to get

$M = \text{antilog}(0.8175) = 6.561 + 0.008 = 6.569$

Therefore, $\log(6.569) = 0.8175$

## Example 1

Find the antilog of the following numbers.

**a)** 0.9335 **b)** 3.0913 **c)** -2.1202

**Solution:**

**a)** To find antilog(0.9335). That is to find a number $M$ such that $\log M = 0.9335$.

**Step 1**. Separate the number 0.9335 as 0.93, 3 and 5

**Step 2**. From the antilogarithm part read the number at the intersection of row 0.93 and column 3, this gives 8.570

**Step 3**. From the mean difference part, read the number at the intersection of row 0.93 and column 5, this gives 10. We write this as 0.010.

**Step 4**. Add the values obtained in steps 2 and 3 to get

$8.570 + 0.010 = 8.580$, Therefore, $\text{antilog}(0.9335) = 8.580$

**b)** In 3.0913, the characterstic is 3. Therefore aftre finding the antilog of 0.0913, we multiply it by $10^3$.

To find antilog(0.0913),

**Step 1**. Separate the number 0.0913 into 0.09, 1 and 3

**Step 2**. From the antilogarithm part read the number at the intersection of row 0.09 and column 1, this gives 1.233.

**Step 3**. From the mean difference part, read the number at the intersection of row 0.09 and column 3, this gives 1. We write this as 0.001.

**Step 4**. Add the values obtained in steps 2 and 3 to get

$1.233 + 0.001 = 1.234$

**Step 5.** $1.234 \times 10^3 = 1234.00$. Therefore the $\text{antilog}(3.0913) = 1234.00$

**c)** To find the antilog of negative numbers like $-2.1202$, we rewrite it as

$$-2.1202 = -2 - 0.1202 + (1 - 1) = (-2 - 1) + (1 - 0.1202)$$
$$= -3 + 0.8798.$$

Then after finding the antilog of 0.8798, we multiply it by $10^{-3}$ to get the antilog(-2.1202)

Now, to find antilog(0.8798)

**Step 1**. Separate the number 0.8798 into 0.87, 9 and 8.

**Step 2**. From the antilogarithm part read the number at the intersection of row 0.87 and column 9 , this gives 7.568.

**Step 3**. From the mean difference part, read the number at the intersection of row 0.87 and column 8, this gives 14. We write this as 0.014.

**Step 4**. Add the values obtained in steps 2 and 3 to get

$7.568 + 0.014 = 7.582$

**Step 5.** $7.582 \times 10^{-3} = 0.007582$.

Therefore the antilog$(-2.1202) = 0.007582$

## Exercise 3.15

Find

a) antilog 0.7412 b) antilog 0.9330 c) antilog 0.9996
d) antilog 0.7 e) antilog 1.3010 f) antilog 0.9953
g) antilog 5.721 h) antilog $(-0.2)$

## Computation with Logarithms

In this section, you will see how logarithms are used for computations of numbers like $\frac{267\times 3252}{403}$, $\sqrt{254}$, etc. Specially common logarithms are used in mathematical computations.

In order to compute a given number $M$, you can perform the following steps:

**Step 1.** Find $\log M$, using the laws of logarithms.

**Step 2.** Find the antilogarithm of $\log M$.

## Example 2

Approximate the values of the following using logarithm.

**a)** $\frac{267\times3252}{403}$ **b)** $\sqrt{254}$

**Solution**:

**a)** Let $M = \frac{267\times3252}{403}$

$$\text{Log } M = \log\left(\frac{267\times3252}{403}\right)$$

$$= \log 267 + \log 3252 - \log 403$$

$$= (2 + \log 2.67) + (3 + \log 3.252) - (2 + \log 4.03)$$

$$= 3 + \log 2.67 + \log 3.252 - \log 4.03$$

$$= 3 + 0.4265 + 0.5122 - 0.6053 = 3.3334$$

$$\log M = 3.3334$$

$$M = \text{antilog}(3.3334) = 2155$$

**b)** Let $M = \sqrt{254}$

$$\log M = \log(\sqrt{254}\,) = \frac{1}{2}\log 254 = \frac{1}{2}(2 + \log 2.54)$$

$$= 1 + \frac{1}{2}\log 2.54 = 1 + \frac{1}{2}(0.4048) = 1.2024$$

$$M = \text{antilog}(1.2024) = 15.93$$

## Exercise 3.16

Compute using logarithms.

a) $4.26 \times 5.73$ b) $\sqrt[5]{25}$ c) $3^{1.42}$

d) $(4.2)^{1.3} \times (0.21)^{4.1}$ e) $\frac{\sqrt{488}}{(2.81)^2}$ f) $\sqrt[5]{0.0461}$

# 3.2 The Exponential Functions and Their Graphs

## 3.2.1 Exponential Functions

### Activity 3.5

Consider a single bacterium which divides every hour.

a) Find the number of bacteria after one hour, two hours, three hours, four hours, and $t$ hours.

b) Complete the following table.

| Time in hour ($t$) | 0 | 1 | 2 | 3 | 4 | … | $t$ |
|---|---|---|---|---|---|---|---|
| Number of bacteria | 1 | | | | | | |

c) Write a formula to calculate the number of bacteria after $t$ hours.

### Definition 3.8

The exponential function $f$ with base $a$ is denoted by $f(x) = a^x$ where $a > 0, a \neq 1$ and $x$ is any real number.

### Example 1

Given $f(x) = 3^x$. Evaluate the following.

a) $f(2)$ b) $f(0)$ c) $f(-1)$

**Solution:**

a) $f(2) = 3^2 = 9.$ b) $f(0) = 3^0 = 1.$ c) $f(-1) = 3^{-1} = \frac{1}{3}.$

### Example 2

Write each of the following functions in the form $f(x) = 2^{kx}$ or $f(x) = 3^{kx}$ for a suitable constant $k$.

a) $f(x) = 4^x$ b) $f(x) = \sqrt{2}^x$ c) $f(x) = \left(\frac{1}{9}\right)^x$

**Solution:**

a) $f(x) = 4^x = (2^2)^x = 2^{2x}$ b) $f(x) = \sqrt{2}^x = \left(2^{\frac{1}{2}}\right)^x = 2^{\frac{1}{2}x}$

c) $f(x) = \left(\frac{1}{9}\right)^x = \left(\frac{1}{3^2}\right)^x = (3^{-2})^x = 3^{-2x}$

**Note**

1) In the definition of exponential function, $a \neq 1$ because if $a = 1$, $f(x) = 1^x = 1$ is a constant function.

2) The exponential function $f(x) = a^x$, where $a > 0, a \neq 1$ is different from all the functions that you have studied in the previous chapters because the variable $x$ is an exponent.

3) A distinct characteristic of an exponential function $f(x) = a^x$ is showing a rapid increase as $x$ increases for $a > 1$ and showing a rapid decrease as $x$ increases for $a < 1$.

4) Many real-life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions.

## Exercise 3.17

1. Given $f(x) = \left(\frac{1}{4}\right)^x$. Find the values of

   a) $f(2)$ b) $f(-2)$ c) $f\left(\frac{1}{2}\right)$ d) $f\left(-\frac{1}{2}\right)$

2. Write each of the following functions in the form $f(x) = 2^{kx}$ or $f(x) = 3^{kx}$ for a suitable constant $k$.

   a) $f(x) = 8^x$ b) $f(x) = \sqrt{3}^x$ c) $f(x) = \left(\frac{1}{81}\right)^x$

   d) $f(x) = \sqrt[3]{2}^x$ e) $f(x) = \left(\frac{1}{27}\right)^{\frac{x}{3}}$ f) $f(x) = \left(\frac{1}{16}\right)^{\frac{-x}{3}}$

## 3.2.2 Graph of Exponential Functions

Graphs of exponential functions can be drawn by plotting points on the $xy$-plane.

**Example 1**

Draw the graph of the exponential function $f(x) = 2^x$.

**Solution:**

First, we calculate values of $f(x)$ for some integer values of $x$ and prepare a table of these values.

$f(-3)=\frac{1}{8}$, $f(-2)=\frac{1}{4}$, $f(-1)=\frac{1}{2}$, $f(0)=1$, $f(1)=2$, $f(2)=4$, $f(3)=8$

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)=2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |

**Note**

- ✓ The function $f(x)=2^x$ is positive for all values of $x$.
- ✓ As $x$ increases, the value of the function gets larger and larger.
- ✓ As $x$ decreases, the value of the function gets smaller and smaller, approaching zero.

Then we plot the points on the $xy$-plane and join them by a smooth curve as shown in the figure 3.1.

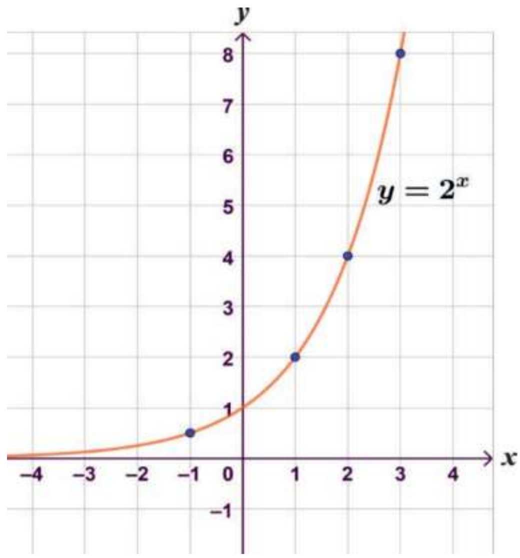



Figure 3.1: Graph of $f(x)=2^x$

Since the domain of a function $y=f(x)$ is the set of all values of $x$ for which the function $f$ is defined and its range is the set of all values of $y$, **the domain of $f(x)=2^x$ is the set of all real numbers and its range is the set of positive real numbers.**

## Activity 3.6

1) For which values of $x$ is $f(x) = 2^x > 1$?
2) For which values of $x$ is $f(x) = 2^x < 1$?
3) Does $f(x) = 2^x$ increase as $x$ increases?
4) What happens to the graph of $f(x) = 2^x$ as $x$ gets larger and larger without bound?
5) What happens to the graph of $f(x) = 2^x$ when $x$ is negative and $|x|$ is very large?
6) Is there a line that the graph of $f(x) = 2^x$ approaches but never touches when $x$ is negative and $|x|$ is very large? What is that line?

## Exercise 3.18

For the function $f(x) = 3^x$,

a) Complete the table of values below.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y = f(x)$ | | | | | |

b) Find the intercepts.

c) Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve.

d) Find the domain and range of $f$.

## Example 2

Draw the graph of the exponential function $f(x) = \left(\frac{1}{2}\right)^x$.

### Solution:

First, we calculate values of $f(x)$ for some integer $x$ and prepare a table of these values.

$f(-3) = 8, f(-2) = 4, f(-1) = 2, f(0) = 1, f(1) = \frac{1}{2}, f(2) = \frac{1}{4}, f(3) = \frac{1}{8}$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = (\frac{1}{2})^x$ | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

**Note**

- The function $f(x) = \left(\frac{1}{2}\right)^x$ is positive for all values of $x$.
- As $x$ increases, the value of the function gets smaller and smaller, approaching zero.
- As $x$ decreases, the value of the function gets larger and larger.

Then we plot the points on the $xy$-plane and join them by a smooth curve as shown in the figure 3.2.

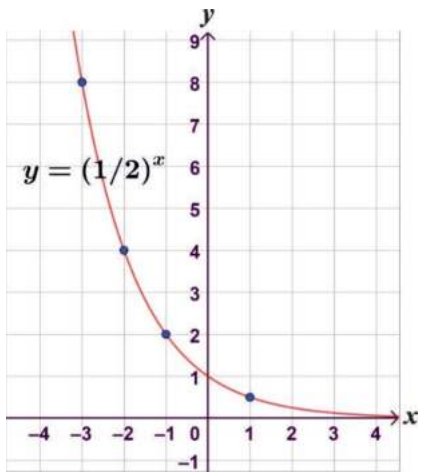



Figure 3.2: Graph of $f(x) = \left(\frac{1}{2}\right)^x$

## Exercise 3.19

For the function $f(x) = \left(\frac{1}{3}\right)^x$,

a) Complete the table of values below.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|
| $y = f(x)$ | | | | | | |

b) Find the intercepts.

c) Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve.

d) Find the domain and range of $f$.

---

The exponential function $f(x) = a^x$, $a > 0$ and $a \neq 1$ has domain of the set of all real numbers and range of the set of all positive real numbers. The $x$-axis (the line $y = 0$) is a horizontal asymptote of $f$. The graph of $f$ has one of the following shapes.

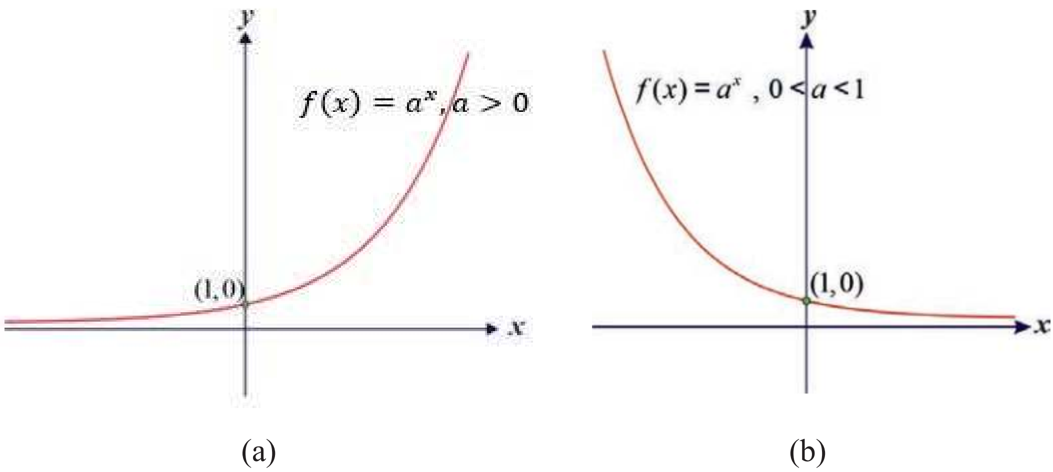



Figure 3.3

The basic characterstics of the exponential function are summarized below.

**Characterstics of Graph of $f(x) = a^x$, $a > 1$**

**a)** Domain: $\mathbb{R}$ = The set of all real numbers.

**b)** Range: $\mathbb{R}^+$ =The set of all positive real numbers.

**c)** $y$-intercept: The point $(0, 1)$.

**d)** Has no $x$-intercept.

**e)** It is increasing on $\mathbb{R} = (-\infty, \infty)$.

**f)** The graph goes upward without bound as $x$ gets larger and positive.

**g)** The graph gets closer to the negative $x$-axis when $x$ is negative and àxà is large.

**h)** Horizontal asymptote: The $x$-axis (the line $y = 0$) is a horizontal asymptote. (See figure 3.3a)

**Characterstics of Graph of $f(x) = a^x$ , $0 < a < 1$.**

**a)** Domain: $\mathbb{R}$ = The set of all real numbers.

**b)** Range: $\mathbb{R}^+$ =The set of all positive real numbers.

**c)** $y$-intercept: The point $(0, 1)$.

**d)** Has no x intercept.

**e)** It is decreasing function. The value of $f$ decreases whenever the value of $x$ increases.

**f)** The graph goes upward without bound when $x$ is negative and $|x|$ is large

**g)** The graph gets closer to the positive $x$-axis when $x$ gets larger and positive.

**h)** Horizontal asymptote: The $x$-axis (the line $y = 0$) is a horizontal asymptote. (See figure 3.3b)

## Example 3

Find the exponential function $f(x) = a^x$ whose graph is given by figure 3.4.

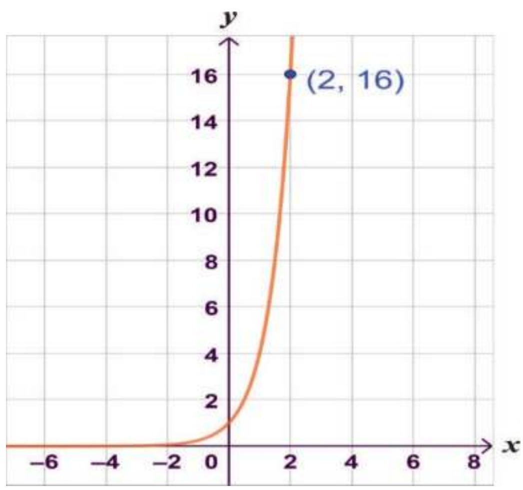



Figure 3.4

**Solution:**

If $f(2) = a^2 = 16$ then $a = 4$. So, $f(x) = 4^x$ is the required function.

## Exercise 3.20

1) Find the exponential function $f(x) = a^x$ whose graph is given by figure 3.5.

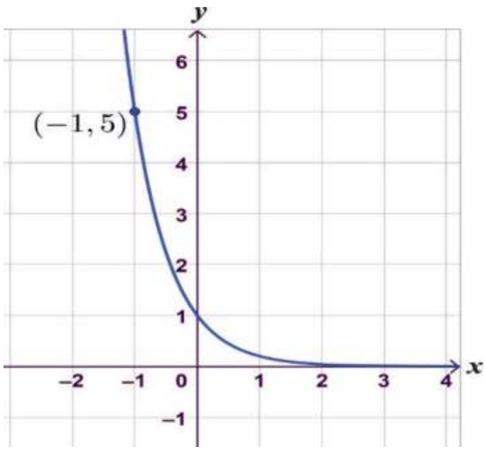



Figure 3.5

2) Using the same coordinate system, draw the graphs of

a) $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

b) $f(x) = 4^x$ and $g(x) = \left(\frac{1}{4}\right)^x$.

## 3.2.3 The Natural Exponential Function

Any positive number can be used as the base for an exponential function, but for the bases the number denoted by the letter $e$ and 10 are used more frequently. The number $e$ is the most important base and convenient for certain applications.

The number $e$ is defined as the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as $n$ becomes large.

The table below shows the values of the expression $\left(1 + \frac{1}{n}\right)^n$ for increasingly large values of $n$. It appears that $e \cong 2.71828$ correct to five decimal places.

| $n$ | $(1+\frac{1}{n})^n$ |
|---|---|
| 1 | 2.00000 |
| 2 | 2.25000 |
| 5 | 2.48832 |
| 10 | 2.59374 |
| 100 | 2.70481 |
| 1000 | 2.71692 |
| 10,000 | 2.71815 |
| 100,000 | 2.71827 |
| 1,000,000 | 2.71828 |

The approximate value to 20 decimal places is $e \cong 2.71828182845904523536$.

## Definition 3.9

The natural exponential function is the exponential function $f(x) = e^x$ with base $e$.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $g(x) = 2^x$ and $h(x) = 3^x$ as shown in the figure 3.6.

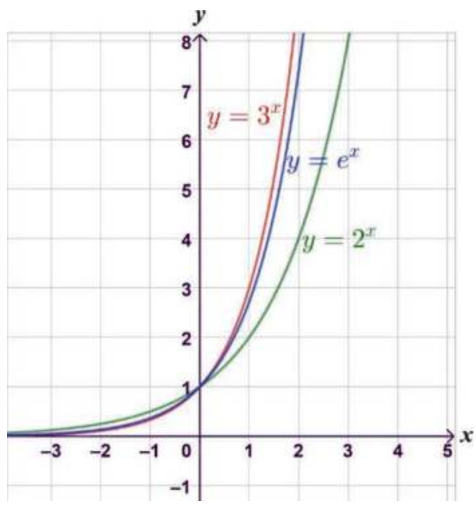


Figure 3.6

## Example 1

Use a scientific calculator to evaluate each expression correct to five decimal places.

**a)** $e^2$ **b)** $e^{-1}$ **c)** $e^{3.5}$

### Solution:

We use the $[e]$ key on a scientific calculator to evaluate the exponential expressions.

**a)** $e^2 \cong 7.38906.$

**b)** $e^{-1} \cong 0.36788.$

**c)** $e^{3.5} \cong 33.11545.$

## Example 2

Construct table of values for some integer values of $x$, sketch the graphs, find the $x$-intercept and $y$-intercept, find the asymptote and give the domain and the range of $y = -e^x$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y = -e^x$ | $-0.14$ | $-0.37$ | $-1$ | $-2.72$ | $-7.39$ |

No $x$-intercept.

The $y$-intercept is (0, -1).

The asymptote is the line $y = 0$ ($x$-axis).

The domain is the set of all real numbers.

The range is the set of negative real numbers.

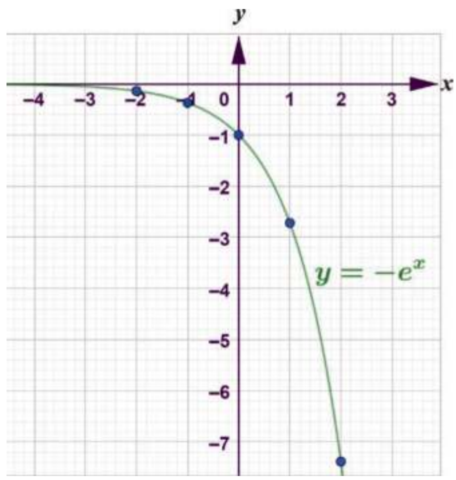



Figure 3.7

## Exercise 3.21

Construct table of values for some integer values of $x$, sketch the graphs, find the $x$-intercept and $y$-intercept, find the asymptote and give the domain and the range of the following functions.

**a)** $y = e^{-x}$ **b)** $y = -e^{-x}$ **c)** $y = 1 + e^{x}$

# 3.3 The Logarithmic Functions and Their Graphs

## 3.3.1 The Logarithmic Functions

Every exponential function $f(x) = a^x$ with $a > 0$ and $a \neq 1$ is a one-to-one function and hence it has an inverse function. The inverse function $f^{-1}$ is called the logarithmic function with base $a$ and is denoted by $g(x) = \log_a x$ where $g = f^{-1}$. This leads us to the following definition of the logarithmic function.

Let $a > 0$ and $a \neq 1$. The logarithmic function with base $a$ denoted by $y = \log_a x$ is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

When evaluating logarithms, remember that a logarithm is an exponent. That is, $\log_a x$ is the exponent to which $a$ must be raised to obtain $x$.

### Example 1

Use the definition of logarithmic function to evaluate each logarithm function at the indicated value of $x$.

**a)** $f(x) = \log_2 x, \quad x = 4$ **b)** $f(x) = \log_4 x, \quad x = 1$

**c)** $f(x) = \log_3 x, \quad x = 27$ **d)** $f(x) = \log_{10} x, \quad x = \frac{1}{10}$

**Solution:**

**a)** $f(4) = \log_2 4 = 2$ because $2^2 = 4$.

**b)** $f(1) = \log_4 1 = 0$ because $4^0 = 1$.

**c)** $f(27) = \log_3 27 = 3$ because $3^3 = 27$.

**d)** $f\left(\frac{1}{10}\right) = \log_{10}\left(\frac{1}{10}\right) = -1$ because $10^{-1} = \frac{1}{10}$.

### Example 2

Write each of the following functions in the form $f(x) = k\log_2 x$ or $f(x) = k\log_3 x$ for a suitable constant $k$.

**a)** $f(x) = \log_4 x$ **b)** $f(x) = \log_9 x$

**c)** $f(x) = \log_{\left(\frac{1}{4}\right)} x$ **d)** $f(x) = \log_{\sqrt{3}} x$

**Solution:**

**a)** $f(x) = \log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{\log_2 x}{\log_2 2^2} = \frac{\log_2 x}{2\log_2 2} = \frac{\log_2 x}{2} = \frac{1}{2}\log_2 x$

**b)** $f(x) = \log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{\log_3 3^2} = \frac{\log_3 x}{2\log_3 3} = \frac{\log_3 x}{2} = \frac{1}{2}\log_3 x$

**c)** $f(x) = \log_{\left(\frac{1}{4}\right)} x = \frac{\log_2 x}{\log_2 2^{-2}} = \frac{\log_2 x}{-2\log_2 2} = -\frac{1}{2}\log_2 x$

**d)** $f(x) = \log_{\sqrt{3}} x = \frac{\log_3 x}{\log_3 \sqrt{3}} = \frac{\log_3 x}{\frac{1}{2}\log_3 3} = 2\log_3 x$

### Example 3

Write each of the following functions in the form $f(x) = c\log_2 kx$ or $f(x) = c\log_5 kx$ for suitable constants $c$ and $k$.

**a.** $f(x) = \log_{\left(\frac{1}{8}\right)}\left(\frac{-x}{2}\right)$ **b.** $f(x) = \log_{\left(\frac{1}{25}\right)}\left(\frac{x}{5}\right)$

**Solution:**

**a.** $f(x) = \log_{\left(\frac{1}{8}\right)}\left(\frac{-x}{2}\right) = \frac{\log_2\left(\frac{-x}{2}\right)}{\log_2\left(\frac{1}{8}\right)} = \frac{\log_2\left(\frac{-x}{2}\right)}{\log_2(2)^{-3}} = \frac{\log_2\left(\frac{-x}{2}\right)}{-3\log_2 2} = \frac{\log_2\left(-\frac{x}{2}\right)}{-3}$

$= -\frac{1}{3}\log_2\left(-\frac{1}{2}x\right)$

**b.** $f(x) = \log_{\left(\frac{1}{25}\right)}\left(\frac{x}{5}\right) = \frac{\log_5\left(\frac{x}{5}\right)}{\log_5\left(\frac{1}{25}\right)} = \frac{\log_5\left(\frac{x}{5}\right)}{\log_5(5)^{-2}} = \frac{\log_5\left(\frac{x}{5}\right)}{-2\log_5 5} = \frac{\log_5\left(\frac{x}{5}\right)}{-2}$

$= -\frac{1}{2}\log_5\left(\frac{1}{5}x\right)$

## Exercise 3.22

1. Given $f(x) = \log_{\left(\frac{1}{4}\right)} x$. Find the values of

   a. $f(2)$    b. $f(-2)$    c. $f\left(\frac{1}{16}\right)$    d. $f(\sqrt{2})$

2. Write each of the following functions in the form $f(x) = k\log_2 x$ or $f(x) = k\log_3 x$ for a suitable constant $k$.

   a. $f(x) = \log_{16} x$    b. $f(x) = \log_{\sqrt{27}} x$

   c. $f(x) = \log_{\left(\frac{1}{9}\right)} x$    d. $f(x) = \log_{\sqrt[3]{2}} x$

3. Write each of the following functions in the form $f(x) = c\log_2 kx$ or $f(x) = c\log_3 kx$ for suitable constants $c$ and $k$.

   a. $f(x) = \log_{\left(\frac{1}{32}\right)}\left(\frac{-x}{2}\right)$    b. $f(x) = \log_{\left(\frac{1}{27}\right)}\left(\frac{x}{3}\right)$

## 3.3.2 Graphs of Logarithmic Functions

If a one-to-one function $f$ has domain A and range B, then its inverse function $f^{-1}$ has domain B and range A. Since the exponential function $f(x) = a^x$ with $a > 0$ and $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we see that its inverse function $g(x) = \log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$. Note that $g(x) = f^{-1}(x)$.

### Example 1

Draw the graph of the logarithmic function $f(x) = \log_2 x$.

**Solution:**

First, we calculate values of $f(x)$ for some values of $x$ which are powers of 2 so that we can find their logarithms and prepare a table of these values.

$f(8) = \log_2 8 = 3,\quad f(4) = \log_2 4 = 2,\quad f(2) = \log_2 2 = 1,\quad f(1) = \log_2 1 = 0,$

$f\left(\frac{1}{2}\right) = \log_2 \frac{1}{2} = -1,\quad f\left(\frac{1}{4}\right) = \log_2 \frac{1}{4} = -2,\quad f\left(\frac{1}{8}\right) = \log_2 \frac{1}{8} = -3$

| $x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|
| $f(x) = \log_2 x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

Then we plot the points on the $xy$-plane and join them by a smooth curve as shown by figure 3.8

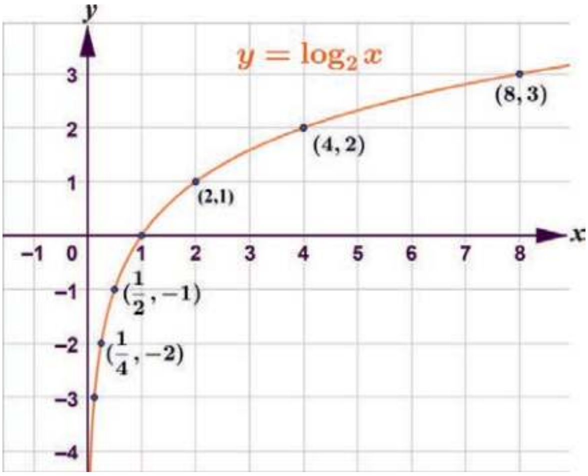



Figure 3.8: Graph of $f(x) = \log_2 x$.

## Exercise 3.23

For the function $f(x) = \log_3 x$

a) Complete the table of values below

| $x$ | $\frac{1}{27}$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 | 27 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | | | | | | | |

b) Find the intercepts.

c) Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve.

d) Find the domain and range of $f$.

Basic characteristics of the graph of $f(x) = \log_a x$ , $a > 1$.

1. Domain: $\mathbb{R}^+$ =The set of all positive real numbers.
2. Range: $\mathbb{R}$ = The set of all real numbers.
3. $x$-intercept: $(1, 0)$
4. It has no y intercept. It does not intersect the $y$-axis.
5. It is increasing function. The value of $f$ increases whenever the value of $x$ increases.
6. The graph goes upward as $x$ gets larger and positive.
7. The graph gets closer to the negative $y$-axis when $x$ gets closer to 0 from the right.

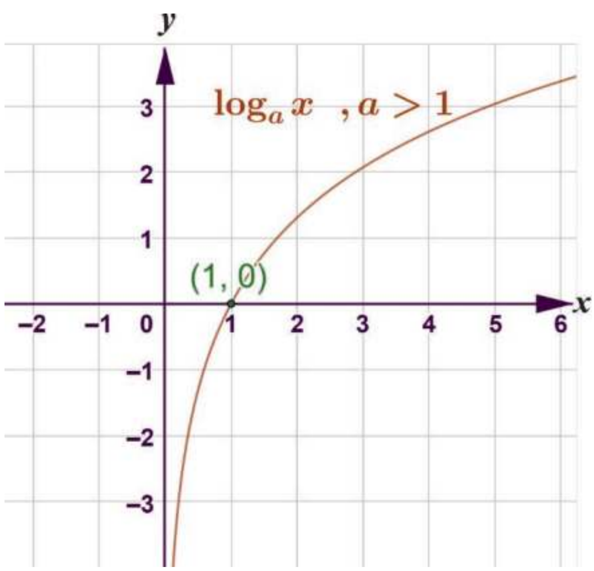



Figure 3.9: Graph of $f(x) = \log_a x$ , $a > 1$

### Example 2

Draw the graph of the logarithmic function $f(x) = \log_{\frac{1}{2}} x$.

**Solution:**

First, we calculate values of $f(x)$ for some values of $x$ which are powers of $\frac{1}{2}$ so that we can find their logarithms and prepare a table of these values.

$f(8) = \log_{\frac{1}{2}} 8 = -3,\quad f(4) = \log_{\frac{1}{2}} 4 = -2,\quad f(2) = \log_{\frac{1}{2}} 2 = -1,$

$f(1) = \log_{\frac{1}{2}} 1 = 0$

$f\left(\frac{1}{2}\right) = \log_{\frac{1}{2}} \frac{1}{2} = 1,\quad f\left(\frac{1}{4}\right) = \log_{\frac{1}{2}} \frac{1}{4} = 2,\quad f\left(\frac{1}{8}\right) = \log_{\frac{1}{2}} \frac{1}{8} = 3$

| $x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|
| $f(x) = \log_{\frac{1}{2}} x$ | 3 | 2 | 1 | 0 | −1 | −2 | −3 |

Then we plot the points on the $xy$-plane and join them by a smooth curve as shown in figure 3.10.

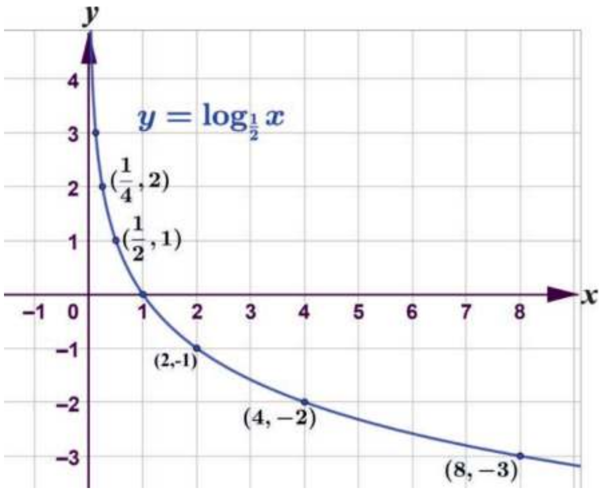



Figure 3.10: Graph of $f(x) = \log_{\frac{1}{2}} x$

## Exercise 3.24

For the function $f(x) = \log_{\left(\frac{1}{3}\right)} x$

a) Complete the table of values below.

| $x$ | $\frac{1}{27}$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 | 27 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | | | | | | | |

b) Find the intercepts.

c) Sketch the graph of $f$, first by plotting the points $(x, y)$ and then joining them by a smooth curve.

d) Find the domain and range of $f$.

Basic characteristics of the graph of $f(x) = \log_a x$ ,$0 < a < 1$.

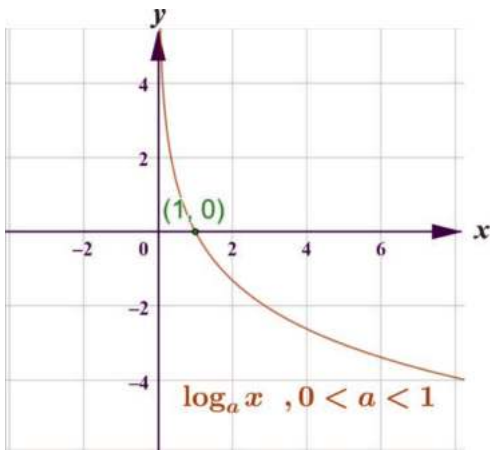



Figure 3.11 Graph of $f(x) = \log_a x$ ,$0 < a < 1$

1. Domain: $\mathbb{R}^+$ =The set of all positive real numbers.
2. Range: $\mathbb{R}$ = The set of all real numbers.
3. $x$-intercept: $(1, 0)$
4. It has no y intercept. It does not intersect the $y$-axis.
5. It is decreasing function. The value of $f$ decreases whenever the value of $x$ increases.
6. The graph goes downward as $x$ gets larger and positive.
7. The graph gets closer to the positive $y$-axis when $x$ gets closer to 0 from the right.

**Note**

For the logarithmic function $f(x) = \log_a x$

- The graph of $-f(x)$ is the reflection of the graph of $f(x)$ along the $x$-axis.
- The graph of $f(-x)$ is the reflection of the graph of $f(x)$ along the $y$-axis.

## Example 1

Sketch the graph of the following logarithmic functions by reflecting the graph of $f(x) = \log_2 x$ either along the $x$-axis or along the $y$-axis.

**a)** $g(x) = -\log_2 x$.

**b)** $h(x) = \log_2(-x)$.

### Solution:

Let $f(x) = \log_2 x$

**a)** Since $g(x) = -\log_2 x = -f(x)$, the graph of $g(x)$ is a reflection of the graph of $f(x)$ along the $x$-axis as shown in the figure 3.12.

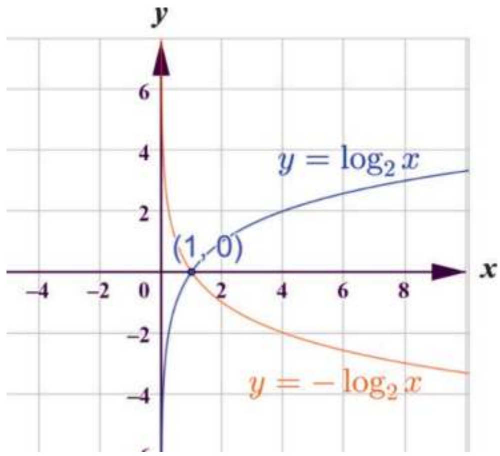



Figure 3.12: Graph of $f(x) = \log_2 x$ *and* $g(x) = -\log_2 x$

**b)** Since $h(x) = \log_2(-x) = f(-x)$, the graph of $h(x)$ is a reflection of the graph of $f(x)$ along the $y$-axis as shown in the figure 3.13.

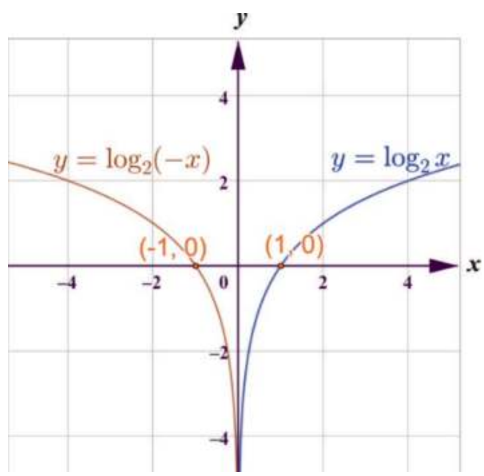



Figure 3.13: Graph of $f(x) = \log_2 x$ and $h(x) = \log_2(-x)$

## Exercise 3.25

Sketch the graph of the following logarithmic functions by reflecting the graph of $f(x) = \log_{\left(\frac{1}{2}\right)} x$ either along the $x$-axis or along the $y$-axis.

**a)** $g(x) = -\log_{\left(\frac{1}{2}\right)} x$

**b)** $h(x) = \log_{\left(\frac{1}{2}\right)}(-x)$

## 3.3.3 Natural Logarithms

### Definition 3.10

The logarithm of a number to the base $e$ is called **natural logarithm** and it is written as

$$\log_e x = \ln x.$$

### Example 1

Find the value of each of the following natural logarithms.

**a)** $\ln 1$ **b)** $\ln e$ **c)** $\ln e^3$

**d)** $\ln \sqrt[3]{e}$ **e)** $\ln \frac{1}{e}$

**Solution:**

**a)** $\ln 1 = \log_e 1 = 0$

**b)** $\ln e = \log_e e = 1$

**c)** $\ln e^3 = 3 \ln e = 3$

**d)** $\ln \sqrt[3]{e} = \ln e^{\frac{1}{3}} = \frac{1}{3} \ln e = \frac{1}{3}$

**e)** $\ln \frac{1}{e} = \ln e^{-1} = -\ln e = -1$

The graphs of $y = e^x$, $y = \ln x$ and the line $y = x$ are shown in figure 3.14.

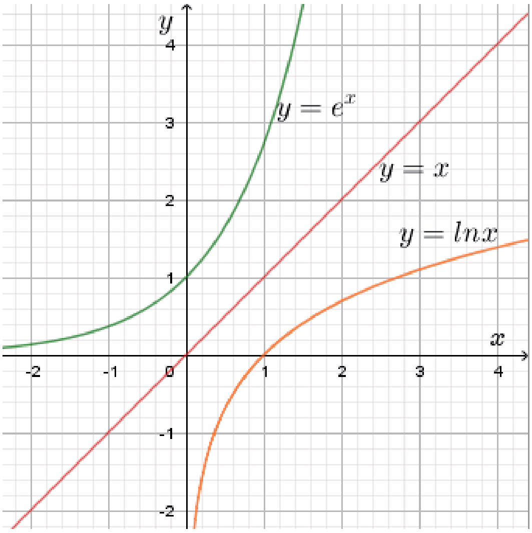



Figure 3.14: Graph of $y = e^x, y = x$ and $y = \ln x$

## Exercise 3.26

Find the values of:

a) $\ln\sqrt[5]{e}$ b) $\ln\left(\frac{1}{e^3}\right)$ c) $e^{\ln 5}$ d) $\ln\sqrt[3]{e^2}$

e) $\ln(e \times e^2)$ f) $\ln(e^x \cdot e^y)$ g) $\ln\left(\frac{e^x}{e^y}\right)$

# 3.4 Solving Exponential and Logarithmic Equations

## 3.4.1 Solving Exponential Equations

An equation in which the variable occurs in the exponent is called an exponential equation. For instance,

$2^x = 16$ and $3^{5x} = 81$ are exponential equations.

To solve exponential equations, we follow the following 3 step procedure.

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of both sides, then use the laws of logarithms, (power law of logarithms) to "bring down the exponent."
3. Solve for the variable.

Moreover, we use the following property:

> **Base-exponent property**
>
> For any real numbers $x, y,\ a > 0,\ a \neq 1,\ a^x = a^y$ if and only if $x = y$.

### Example 1

Solve $2^{(2x-1)} = 8$.

**Solution:**

Since $8 = 2^3$, we have

$$2^{(2x-1)} = 8 = 2^3$$

$$2^{(2x-1)} = 2^3$$

$$2x - 1 = 3$$

$$x = 2$$

So, the solution is $x = 2$.

Geometrically, the solution $x = 2$, is the first coordinate of the point of intersection of the graphs $y = 2^{(2x-1)}$ and $y = 8$ as shown in the figure (Figure 3.15).

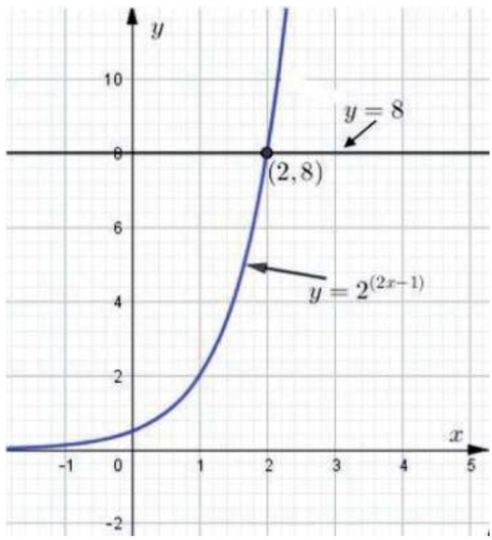



Figure 3.15

### Example 2

Solve $2^{x^2-3x} = 16$.

**Solution:**

Since $16 = 2^4$, we have $2^{x^2-3x} = 2^4$

$$x^2 - 3x = 4$$

$$x^2 - 3x - 4 = 0$$

$$(x + 1)(x - 4) = 0$$

So, the solutions are $x = -1$ and $x = 4$.

## Exercise 3.27

Solve for $x$.

a) $5^x = 125$ b) $3^{3-x} = 81$ c) $4^{2x-5} = 2^{3x+6}$

d) $\frac{1}{8} = \left(\frac{1}{4}\right)^x$ e) $2^{-x} = 512$ f) $3^{x^2-2} = 9$

g) $7^{x^2+x} = 49$ h) $3^{3(x+2)} = 9^{x+2}$ i) $3\left(\frac{27}{8}\right)^{\frac{2}{3}x} = 2\left(\frac{32}{243}\right)^{2x}$

### Example 3

Using logarithm find $x$ if $2^x = 20$.

**Solution:**

Taking the common logarithm on both sides, we obtain:

$$\log 2^x = \log 20$$

$$x \log 2 = \log(2 \times 10) \quad \text{power law of logarithm}$$

$$x \log 2 = \log 2 + \log 10 \quad \text{product law of logarithm}$$

$$x = \frac{\log 2 + 1}{\log 2} \quad \text{since } \log 10 = 1 \text{ and dividing each side by } \log 2$$

But, from the common logarithm table, $\log 2 \approx 0.3010$.

Hence, $x \approx \frac{0.3010+1}{0.3010}$

$\approx \frac{1.3010}{0.3010}$

$\approx 4.32226$

So, the solution is about 4.32226.

Geometrically, the solution $x = 4.32226$ is the first coordinate of the point of intersection of the graphs $y = 2^x$ and $y = 20$ as shown in the figure 3.16.

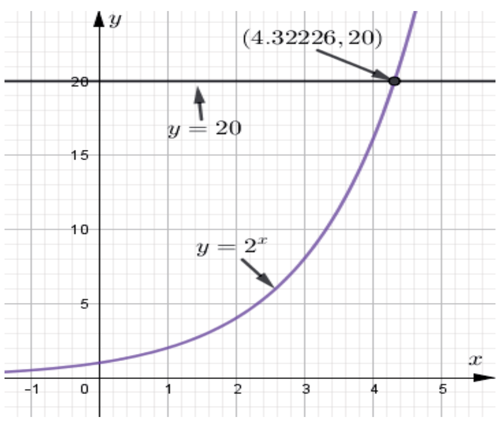



Figure 3.16

## Exercise 3.28

Solve for $x$ by taking the common logarithm of each side.

a) $3^x = 15$ b) $10^x = 13.4$ c) $10^{2x+1} = 92$

d) $(6)^{3x} = 5$ e) $4^{2x} = 61$ f) $(1.05)^x = 2$

g) $10^{5x-2} = 348$ h) $3^x = 0.283$ i) $2^x = 0.283$

## 3.4.2 Solving Logarithmic Equations

A logarithmic equation is **an equation that involves the logarithm of an expression containing a variable**. For instance, $\log_2(x+3) = 4$ is logarithmic equation.

Since the logarithm of non-positive numbers does not exist, before trying to find the solution of $\log_2(x+3) = 4$, you have to restrict $x$ such that $x + 3 > 0$. That is, $x > -3$. The set of all numbers greater than $-3$ is called the **universal set** or simply the **universe** of the equation $\log_2(x+3) = 4$.

We use the following property to solve logarithmic equations.

For any positive real numbers $x, y, a > 0$ and $a \neq 1$

$\log_a x = \log_a y$ if and only if $x = y$.

We use the following procedures to solve logarithmic equations.

1. State the universe.
2. Collect the logarithmic term on one side of the equation.
3. Write the equation in exponential form.
4. Solve for the variable.

### Example 1

Solve the logarithmic equation $\log_2(x+3) = 4$.

**Solution:**

If $x + 3 > 0$ then the universe is $x > -3$.

$$\log_2(x + 3) = 4$$

$x + 3 = 2^4$ exponential form

$x + 3 = 16$

$x = 13 > -3$

So, the solution is $x = 13$.

Geometrically, the solution $x = 13$, is the first coordinate of the point of intersection of the graphs $y = \log_2(x + 3)$ and $y = 4$ as shown in the figure 3.17.

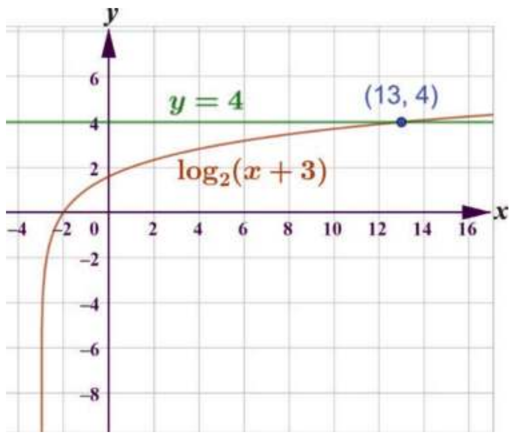



Figure 3.17: Graph of $y = \log_2(x + 3)$

## Example 2

Solve $\log_3(5x - 2) = 2 - \log_3 4$.

**Solution**:

If $5x - 2 > 0$ then $x > \frac{2}{5}$.

$\log_3(5x - 2) = 2 - \log_3 4$

$\log_3(5x - 2) + \log_3 4 = 2$

$\log_3 4(5x - 2) = 2$ Product law

$\log_3(20x - 8) = 2$

$20x - 8 = 3^2 = 9$

$x = \frac{17}{20} > \frac{2}{5}$

$x = \frac{17}{20}$ is the solution.

## Exercise 3.29

State the universe and solve for $x$.

a) $\log_2(3x - 1) = 5$

b) $\log_{\sqrt{2}} x = 6$

c) $\log_2(x^2 - 3x) = 4$

d) $\log_2(x - 1) + \log_2 3 = 3$

e) $\log(x^2 - 121) - \log(x + 11) = 1$

f) $\log_x(x + 6) = 2$

g) $\log x - \log 3 = \log 4 - \log(x + 4)$

h) $\log_2\left(1 + \frac{1}{x}\right) = 3$

i) $\log_2 2 + \log_2(x + 2) - \log_2(3x - 5) = 3$

### Example 1

Using logarithm find $x$ if $3^{x+4} = 2^{-x}$.

**Solution:**

Taking the common logarithm on both sides, we obtain:

$$\log 3^{x+4} = \log 2^{-x}$$
$$(x + 4)\log 3 = -x\log 2$$
$$x\log 3 + 4\log 3 = -x\log 2$$
$$x\log 3 + x\log 2 = -4\log 3$$
$$x(\log 3 + \log 2) = -4\log 3$$
$$x = \frac{-4\log 3}{\log 3 + \log 2}$$

But, from the common logarithm table, we have

$$\log 2 \cong 0.3010\,;\ \log 3 \cong 0.4771.$$

Hence, $x \cong \frac{-4(0.4771)}{0.4771+0.3010}$

$$\cong \frac{-1.9084}{0.7781}$$
$$\cong -2.45264.$$

So, the solution is about $-2.45264$.

Geometrically, the solution $x = -2.45264$ is the first coordinate of the point of intersection of the graphs $y = 3^{x+4}$ and $y = 2^{-x}$ as shown in the figure 3.18.

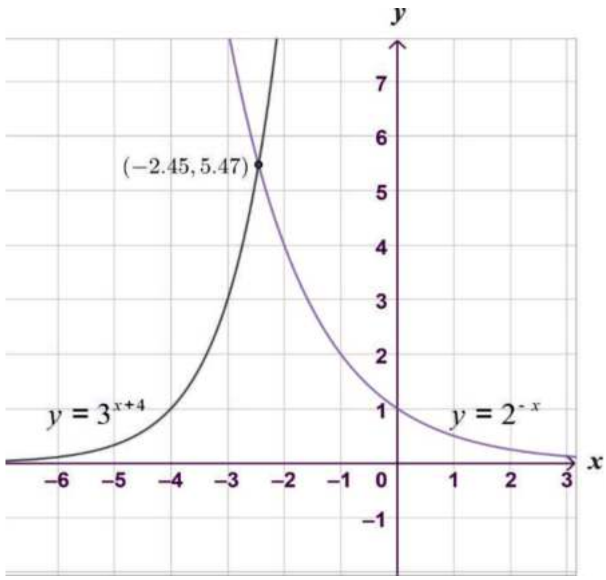



Figure 3.18

**Exercise 3.30**

Using logarithm find $x$ if,

a) $3^{x-1} = 2^x$.    b) $9^x = 8^{x-1}$.

## 3.5 Relation Between Exponential and Logarithmic Functions with the Same Base

Consider the tables of values that we have constructed in the previous section for the exponential function $y = 2^x$ and logarithmic function $y = \log_2 x$ having the same base 2.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |

| $x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|
| $y = \log_2 x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

We see that the values of $x$ and $y$ are interchanged in the functions $y = 2^x$ and $y = \log_2 x$. That is, the domain of $y = 2^x$ is the range of $y = \log_2 x$, the range of $y = 2^x$ is the domain of $y = \log_2 x$ and vice-versa.

Now let us sketch the graphs of both functions on the same co-ordinate system as shown in the figure 3.19.

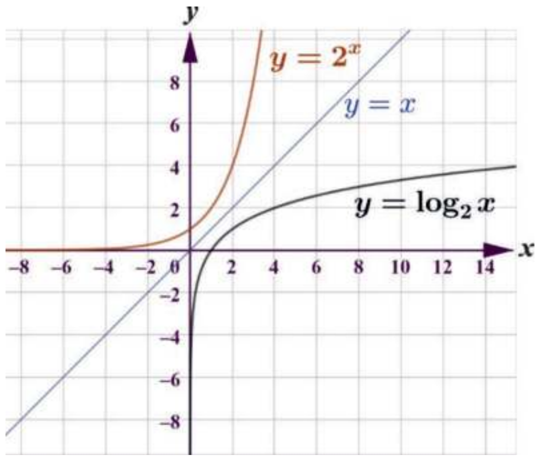



Figure 3.19: Graph of $y = \log_2 x$ and $y = 2^x$

Observe that the graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ along the line $y = x$ as shown in figure 3.19.

Generally, the functions $y = a^x$ and $y = \log_a x$, $a > 1$ are inverses of each other. The relation between the functions $y = a^x$ and $y = \log_a x$, $a > 1$ is shown graphically in figure 3.20.

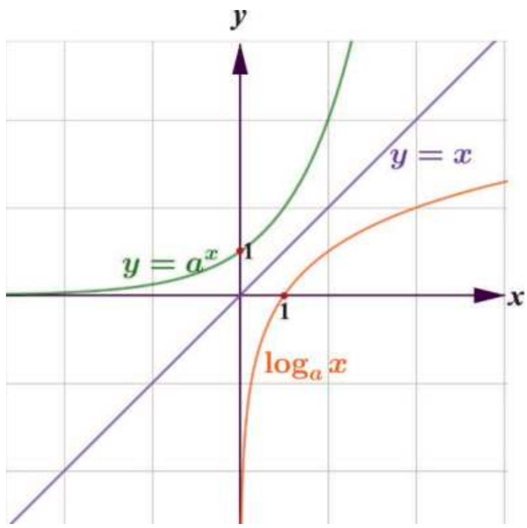



Figure 3.20: Graph of $y = \log_a x$ and $y = a^x$ for $a > 1$

From the graphs above, observe that:

1. The domain of $y = a^x$ is the set of all real numbers, that is the range of

$y = \log_a x$.

2. The range of $y = a^x$ is the set of all positive real numbers, that is the domain of $y = \log_a x$.
   a. Domain of $y = a^x$ = Range of $y = \log_a x$.
   b. Range of $y = a^x$ = Domain of $y = \log_a x$.
3. The $x$-axis is the horizontal asymptote of the graph of $y = a^x$; the $y$-axis is a vertical asymptote of the graph of $y = \log_a x$.
4. The point $(0,1)$ is the $y$-intercept of the graph of $y = a^x$; the point $(1,0)$ is the $x$-intercept of the graph of $y = \log_a x$.

Figure 3.21 shows graphs of the family of logarithmic functions with bases 2, 3, 4, 5 and 10. These graphs can be drawn by reflecting the graphs of $y = 2^x, y = 3^x, y = 4^x, y = 5^x$ and $y = 10^x$ along the line $y = x$.

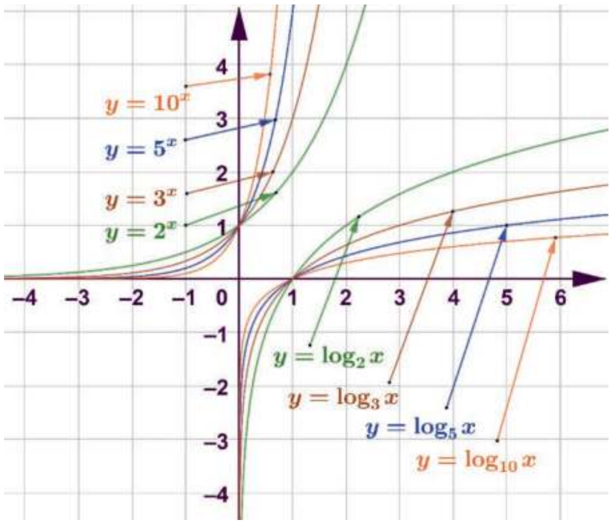



Figure 3.21

**Example 1**

For the exponential function $y = \left(\frac{1}{2}\right)^x$ and the logarithmic function $y = \log_{\frac{1}{2}} x$,

**a.** Complete the table of values below.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = \left(\frac{1}{2}\right)^x$ | | | | | | | |

| $x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|
| $g(x) = \log_{\frac{1}{2}} x$ | | | | | | | |

**b.** Sketch their graphs on the same $xy$-plane.

**c.** Find the domain and the range of the functions.

**d.** State the relation that exists between the domain and the range of the functions.

**Solution:**

**a.**

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x) = \left(\frac{1}{2}\right)^x$ | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

| $x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|
| $g(x) = \log_{\frac{1}{2}} x$ | 3 | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ |

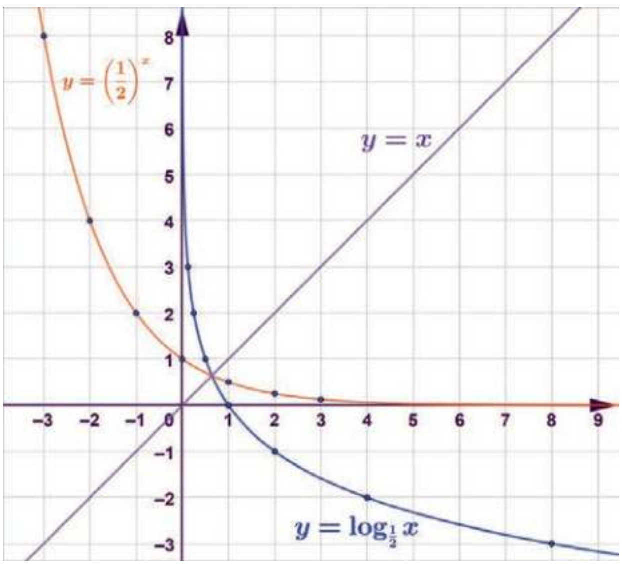



Figure 3.22: Graph of $y = \left(\frac{1}{2}\right)^x$ and $y = \log_{\frac{1}{2}} x$

**c.** The domain of $f(x)$ is the set of all real numbers and its range is the set of positive real numbers. The domain of $g(x)$ is the set of positive real numbers and its range is the set of all real numbers.

**d.** Domain of $f(x)$ = Range of $g(x)$ and Range of $f(x)$ = Domain of $g(x)$.

## Exercise 3.31

Let $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = \log_{\frac{1}{3}} x$.

1. Sketch the graphs of $f(x)$ and $g(x)$ on the same $xy$-plane.
2. Find the domain and the range of $f(x)$.
3. Find the domain and the range of $g(x)$.
4. Compare the domain of $f(x)$ with the range of $g(x)$.
5. Compare the range of $f(x)$ with the domain of $g(x)$.

# 3.6 Applications

## 3.6.1 Compound Interest

Exponential functions occur in evaluating compound interest. Suppose an amount of money $P$, called the principal, is invested at an annual interest rate $r$, compounded once a year, then after a year the interest is $Pr$. If the interest is added to the principal at the end of the year, the new amount $A(1)$ of money is,

$$A(1) = P + Pr = P(1 + r)$$

If the interest is reinvested, then the new principal is $A(1) = P(1 + r)$, and after another year the interest is $rA(1)$, then the amount after the end of the second year, $A(2)$ is

$$A(2) = A(1) + rA(1) = A(1)(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$$

Similarly, at the end of the third year, the amount $A(3)$ is $A(3) = P(1 + r)^3$.

Generally, after the end of $t$ years, the amount $A(t)$ is $A(t) = P(1 + r)^t$.

Observe that this is an exponential function with base $1 + r$.

Let $n$ be the number of compounding per year and $t$ be the number of years. Then the product $nt$ represents the total number of times the interest will be compounded and the interest rate per compounding period is $\frac{r}{n}$. This leads to the following formula indicated as in theorem 3.5 for the amount after $t$ years.

**Theorem 3.5 Compound interest**

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A(t)$= amount after $t$ years,

$P$ = principal,

$r$ = interest rate per year,

$n$ = number of times interest is compounded per year and

$t$ = number of years.

## Example 1

A total of Birr 20000 is invested at an interest rate of 7% per year. Find the amounts in the account after 5 years if the interest is compounded

**a)** annually,

**b)** semi-annually,

**c)** quarterly,

**d)** monthly,

**e)** daily.

**Solution:**

**a)** Here we have $P = 100,\ r = 7\% = 0.07,\ n = 1$ and $t = 5$.

Using the formula for compound interest with $n$ compounding per year, we have $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ . . . Formula for compound interest

$= 20000\left(1 + \frac{0.07}{1}\right)^{1(5)}$ . . . Substitute for $P, r, n$ and $t$

$= 20000(1.07)^5$ . . . Simplifying

$\cong 28051.03$. . . use a calculator

Therefore, the amount in the account after 5 years will be about Birr 28051.03.

**b)** For semi-annually compounding, $n = 2$. Hence, after 5 years at 7% rate, the amount in the account is

$$A(5) = 20000\left(1 + \frac{0.07}{2}\right)^{2(5)} = 20000(1.035)^{10} \cong 28211.98.$$

Therefore, the amount in the account after 5 years will be about Birr 28211.98.

**c)** For quarterly compounding, $n = 4$. Thus, after 5 years at 7% rate, the amount in the account is

$$A(5) = 20000\left(1 + \frac{0.07}{4}\right)^{4(5)} = 20000(1.0175)^{20} \cong 28295.56.$$

Therefore, the amount in the account after 5 years will be about Birr 28295.56.

**d)** For monthly compounding, $n = 12$. So, after 5 years at 7% rate, the amount in the account is

$$A(5) = 20000\left(1 + \frac{0.07}{12}\right)^{12(5)} = 20000(1.00583)^{60} \cong 28352.51.$$

Therefore, the amount in the account after 5 years will be about Birr 28352.51.

**e)** For daily compounding, $n = 365$. Therefore, after 5 years at 7% rate, the amount in the account is

$$A(5) = 20000\left(1 + \frac{0.07}{365}\right)^{365(5)} = 20000(1.00019)^{1825} \cong 28380.40.$$

Therefore, the amount in the account after 5 years will be about Birr 28380.40.

**Note**

The interest paid increases as the number of compounding period $n$ increases.

## Exercise 3.32

Suppose that Birr 10,000 is invested at 7% interest compounded annually.

a. Find the function (formula) for the amount to which the investement grows after $t$ years.

b. Find the amount of money in the account at $t = 1, 5, 10, 15$ and 20 years.

c. Find the time $t$ at which the investement is double.

Let us see what happens as $n$ gets larger and larger without bound.

If we let $m = \frac{n}{r}$, then

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{1}{m}\right)^{mrt} = p\left[\left[\left(1+\frac{1}{m}\right)^{m}\right]\right]^{rt}$$

But, as $m$ becomes larger and larger, the quantity $\left(1+\frac{1}{m}\right)^{m}$ approaches the irrational number $e$.

Therefore, the amount $A$ approaches

$$A = Pe^{rt}$$

This expression gives the amount when the interest is continuously compounded.

Continuously compound interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where $A(t)$ = amount after $t$ years, $P$ = principal, $r$ = interest rate per year and $t$ = number of years.

### Example 2

A total of Birr 100 is invested at an interest rate of 7% per year. Find the amount in the account after 5 years if the interest is compounded continuously.

**Solution:**

For countinuous compounding, the amount in the account after 5 years at 7% rate is

$A(5) = 100(e^{0.07(5)}) = 100(e^{0.35}) = 100(1.41906) \cong 141.91.$

The amount in the account after 5 years is Birr 141.91.

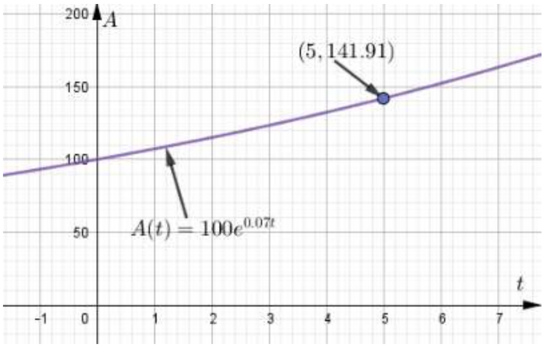



Figure 3.23

## Exercise 3.33

1) If Birr 40,000 is invested in an account for which interest is compounded continuously, find the amount of the investement at the end of 10 years for the following interest rates.

   a. 6%  b. 7%

   c. 6.5%  d. 7.5%

2) Suppose you are offered a job that lasts one month, and you are to be very well paid. Which of the following methods of payment is more profitable for you?

   a) Birr one million at the end of the month.

   b) Two cents on the first day of the month, four cents on the second day, eight cents on the third day, and, in general, $2^n$ cents on the $n^{th}$ day.

## 3.6.2 Population Growth

The exponential function

$P(t) = P_0 e^{kt}, k > 0$ is a mathematical model of many kinds of population growth. In this function, $P_0$ is the population at initial time $t_0$, $P(t)$ is the population after time $t$, and $k$ is called the exponential growth rate. The graph of such an equation is shown in the figure 3.24.

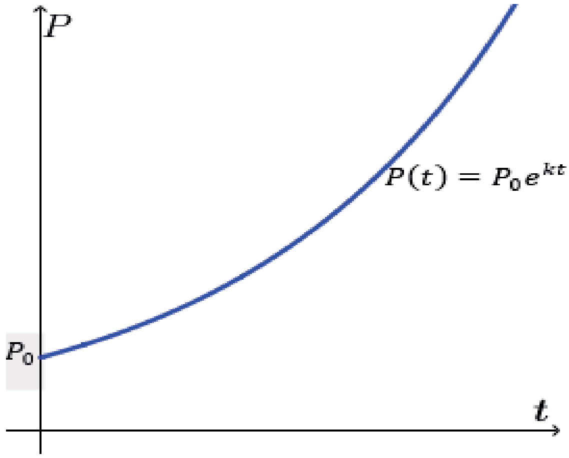



Figure 3.24

**Example 3**

In 2013 E.C, the population of a country was 110 million, and the exponential growth rate was 2.3 % per year.

**a)** Find the exponential growth function.

**b)** Estimate the population in 2018 E.C.

**c)** How many years will it take for the population to be doubled?

**Solution:**

**a)** Here $P_0 = 110$ million, the population in the year 2013 $(t = 0)$ and the growth rate $k = 2.3\% = 0.023$. So, the exponential growth function is:

$$P(t) = (110{,}000{,}000)e^{0.023t}.$$

**b)** In the year 2018, $t = 5$. To find the population in 2018, we substitute 5 for $t$, i.e.,

$$P(5) = (110{,}000{,}000)e^{0.023(5)} = (110{,}000{,}000)e^{0.115} \cong 123{,}406{,}078.$$

Therefore, the population will be 123,406,078 in 2018.

**c)** We find $t$ for which $P(t) = 2P_0 = 2(110{,}000{,}000) = 220{,}000{,}000.$

To find the time, we solve the equation:

$$P(t) = (110{,}000{,}000)e^{0.023t}$$

$$220{,}000{,}000 = (110{,}000{,}000)e^{0.023t}$$

$$2 = e^{0.023t} \quad . . . \text{ dividing each side by 100,000,000}$$

$$\ln 2 = \ln e^{0.023t} \quad . . . \text{ taking the natural logarithm of both sides}$$

$$\ln 2 = 0.023t \underbrace{\ln e}_{=1} \quad . . . \text{ power law of logarithm}$$

$$\ln 2 = 0.023t$$

$$t = \frac{\ln 2}{0.023} \quad . . . \text{ dividing each side by 0.023}$$

$$t \cong \frac{0.69315}{0.023} \quad \text{because } \ln 2 \cong 0.69315$$

$$t \cong 30.14$$

Hence, it takes approximately 30 years for the population of the country to be doubled.

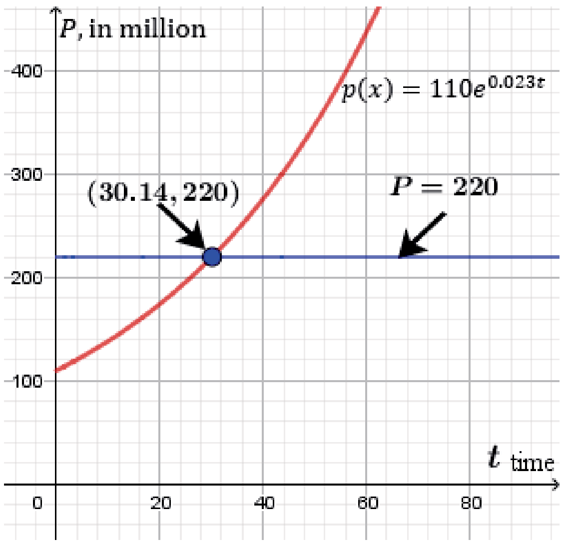



Figure 3.25

From the graphs of $y = 110{,}000{,}000\, e^{0.023\,t}$ and $y = 220{,}000{,}000$ above, we see that the first coordinate of the point of intersection of the graphs is about 30.14.

## Exercise 3.34

A culture contains 10,000 bacteria initially. After an hour, the bacteria count is 25,000.

a. Find the doubling period.

b. Find the number of bacteria after 5 hours

### 3.6.3 The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Soren Peter Lauritz Sorensen, in 1909, proposed a more convenient measure. He defined

$$pH = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M). Solutions with a $pH = 7$ are defined **neutral**, those with $pH < 7$ are **acidic**, and those with $pH > 7$ are **basic**.

**Example 4**

The hydrogen ion concentration of a sample of human blood was measured to be $[\mathrm{H}^+] = 4.53 \times 10^{-8}$ M. Find the $pH$ and determine whether the blood is acidic or basic.

**Solution:**

We have $pH = -\log[\mathrm{H}^+] \;\; = -\log[4.53 \times 10^{-8}]$

$= -[\log(4.53) + \log 10^{-8}]$ . . . product law of logarithm

$= -[0.6561 - 8]$ (from log table $\log(4.53) = 0.6561$)

$= 7.344$

Since $pH = 7.344 > 7$, the blood is basic.

**Example 5**

The most acidic rainfall ever measured occurred in Scotland in 1974 and its $pH$ was 2.4. Find the hydrogen ion concentration.

**Solution:**

$pH = -\log[H^+]$

$2.4 = -\log[H^+]$ $\quad \log[H^+] = -2.4$ … multiply both sides by -1.

$\log[H^+] = (3 - 2.4) - 3$ $\quad$ ($3 - 3 = 0$ ading 0 to a number make no change)

$\log[H^+] = 0.6 + (-3)$

$\text{antilog}(\log[H^+]) = \text{antilog}(0.6 + (-3))$

$[H^+] = 3.981 \times 10^{-3}$

So, the hydrogen ion concentration of the rainfall was about $3.98 \times 10^{-3}\ M$.

## Exercise 3.35

1. The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance and determine whether it is acidic or basic.
   a. Lemon juice: $[H^+] = 5 \times 10^{-3}\ M$
   b. Tomato juice: $[H^+] = 3.2 \times 10^{-4}\ M$
   c. Seawater: $[H^+] = 5 \times 10^{-9}\ M$
2. The $pH$ reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.
   a) Vinegar : $pH = 3$
   b) Milk: $pH = 6.5$

## Summary

1. For a natural number $n$ and a real number $a$, the power $a^n$, read "the $n^{th}$ power of a" or "$a$ raised to $n$", is defined as follows:

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n\, factors}$$

In the symbol $a^n$, $a$ is called **the base** and $n$ is called **the exponent**.

2. Laws of exponenets: For a real number $a$ and natural numbers $m$ and $n$,

   a) $a^m \times a^n = a^{m+n}$.

   b) $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$.

   c) $(a^m)^n = a^{mn}$.

   d) $(a \times b)^n = a^n \times b^n$.

   e) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$.

3. $a^0 = 1, a \neq 0$.

4. Laws/properties of logarithms: For any positive numbers $x$ and $y$ and $a > 0$ and $a \neq 1$,

   a) $\log_a xy = \log_a x + \log_a y$.

   b) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

   c) $\log_a x^r = r\log_a x$.

   d) $\log_a x = \frac{\log_b x}{\log_b a}$.

   e) $\log_a a = 1$.

   f) $\log_a 1 = 0$.

   g) $a^{\log_a x} = x$.

5. The exponential function $f$ with base $a$ is denoted by $f(x) = a^x$, where $a > 0, a \neq 1$, and $x$ is any real number.

**6.** The natural exponential function is $f(x) = e^x$, where $e$ is the constant 2.718281828…. Its graph has the same basic characteristics as the graph of $f(x) = a^x$.

**7.** The graphs of the exponential functions $f(x) = a^x$ and $f(x) = a^{-x}$ have one $y$-intercept (0,1), one horizontal asymptote (the $x$-axis), and are continuos.

**8.** For $x > 0, a > 0, and\ a \neq 1, y = \log_a x$ if and only if $x = a^y$. The function $f(x) = \log_a x$ is called the logarithmic function with base $a$.

**9.** The graph of the logarithmic function $f(x) = \log_a x$ where $a > 1$, is the graph of the inverse of $f(x) = a^x$.

**10.** For $x > 0, y = \ln x$ if and only if $x = e^y$. The function given by $f(x) = \log_e x = \ln x$ is called the natural logarithmic function. Its graph has the same basic characterstics as the graph of $f(x) = \log_a x$. They have the same $x$-intercept (1,0), the same verticall asymptote (the $y$-axis), and are continuos.

## Review Exercise

**1.** $5^3 = 125$, so $\log_{\square} \square = \square$.

**2.** $\text{Log}_5 25 = 2$, so $\square^{\square} = \square$.

**3.** Let $f(x) = \log_4 x$. Then find $f(4)$, $f(1)$, $f\left(\frac{1}{4}\right)$, $f(16)$ and $f(2)$.

**4.** Find the values of the given logarithms.

**a)** $\log_3 3$ **b)** $\log_3 1$ **c)** $\log_3 3^4$

**d)** $\log_{\left(\frac{1}{27}\right)} 3$ **e)** $\log_{\sqrt{2}} 16$ **f)** $\log_4 \left(\frac{1}{4}\right)$

**g)** $\log_{\left(\frac{1}{5}\right)} 5$ **h)** $\log_{\left(\frac{1}{16}\right)} 64$ **i)** $\log_{\left(\frac{1}{3}\right)} 9$

**j)** $\log_5 0.2$ **k)** $\log_{0.001} 0.1$ **l)** $\log_{10} \sqrt{10}$

**m)** $\log_5 20 + \log_5 \left(\frac{125}{4}\right) - \log_5 \left(\frac{1}{25}\right)$

For question numbers 5 and 6, fill in the table by finding the appropriate logarithmic or exponential form of the equation.

**5.**

| Logarithmic form | Exponential form |
|---|---|
| $\log_7 7 = 1$ | |
| $\log_8 64 = 2$ | |
| | $8^{\frac{2}{3}} = 4$ |
| | $8^3 = 512$ |
| $\log_8 \left(\frac{1}{8}\right) = -1$ | |
| | $8^{-2} = \frac{1}{64}$ |

**6.**

| Logarithmic form | Exponential form |
|---|---|
| | $4^3 = 64$ |
| $\log_4 2 = \frac{1}{2}$ | |
| | $4^{\frac{3}{2}} = 8$ |
| $\log_4 (\frac{1}{16}) = -2$ | |
| $\log_4 (\frac{1}{2}) = -\frac{1}{2}$ | |
| | $4^{\frac{-5}{2}} = \frac{1}{32}$ |

**7.** Express the logarithmic statement in to exponential statement.

**a)** $\log_5 125 = 3$ **b)** $\log_5 1 = 0$

**c)** $\log_{10} 0.1 = -1$ **d)** $\log_8 512 = 3$

**e)** $\log_8 2 = \frac{1}{3}$ **f)** $\log_9 3 = \frac{1}{2}$

**g)** $\log_3 81 = 4$ **h)** $\log_2 \frac{1}{8} = -3$

**8.** Express the exponential statement in to logarithmic statement.

**a)** $3^3 = 27$ **b)** $10^{-3} = 0.001$

**c)** $10^3 = 1000$ **d)** $81^{\frac{1}{2}} = 9$

**e)** $8^{-1} = \frac{1}{8}$ **f)** $2^{-3} = \frac{1}{8}$

**g)** $4^{-\frac{3}{2}} = 0.125$ **h)** $10^{-3} = 0.001$

**9.** Use the definition of logarithmic function to find $x$.

**a)** $\log_{\sqrt{2}} x = 6$ **b)** $\log_2 32 = x$

**c)** $\log_5 x = 4$ **d)** $\log_{10} 0.1 = x$

**e)** $\log_{(\frac{1}{2})} 2 = x$ **f)** $\log_4 x = 2$

g) $\log_x 6 = \frac{1}{2}$

h) $\log_x 3 = \frac{1}{3}$

**10.** Given $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$. Then find the following logarithms.

a) $\log_2 \sqrt{3}$

b) $\log_2 0.3$

c) $\log_3 0.6$

d) $\log_2 108$

e) $\log_3 5$

f) $\log_4 75$

**11.** Match the function with its graph.

a) $f(x) = 4^x$

b) $f(x) = \left(\frac{1}{4}\right)^x$

c) $g(x) = \log_4 x$

d) $f(x) = \log_{\frac{1}{4}} x$

I)

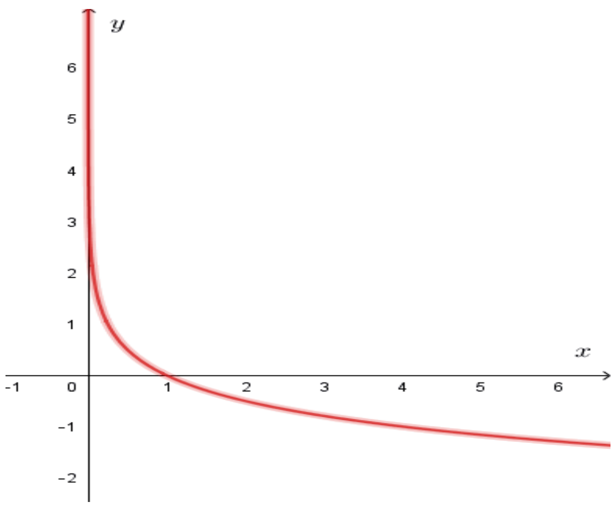



Figure 3.26

**II)**

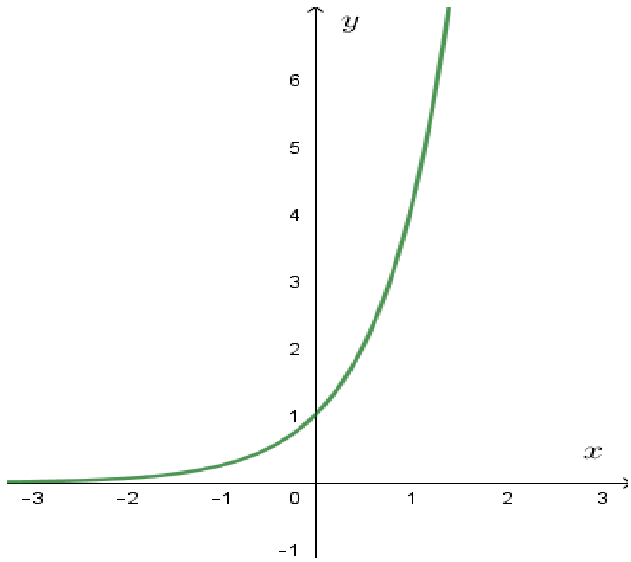



Figure 3.27

**III)**

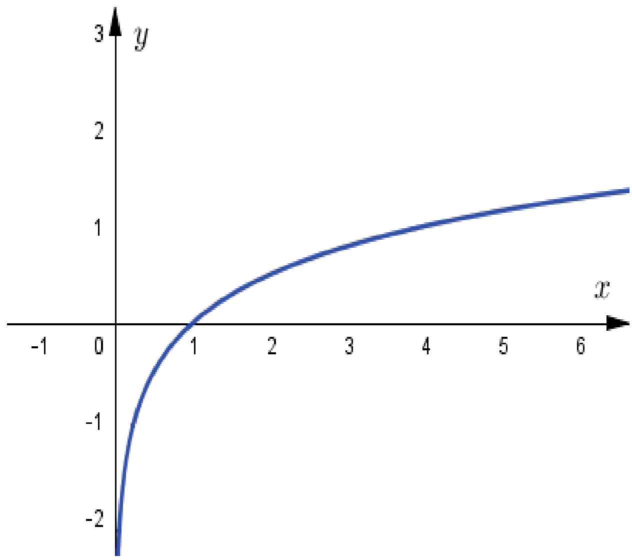



Figure 3.28

**IV)**

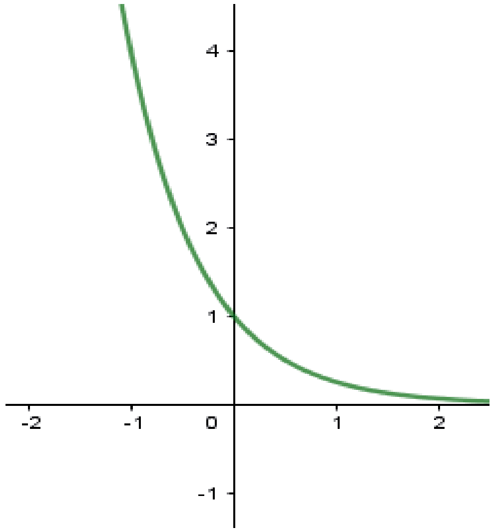



Figure 3.29

**12.** In each of the following, tell which is greater.

**a)** $\sqrt[8]{5}$ or $\sqrt[4]{2}$ **b)** $\left(\frac{1}{3}\right)^{\sqrt{3}}$ or $\left(\frac{1}{3}\right)^{3}$

**c)** $(\sqrt{0.2})^{-3.5}$ or 1 **d)** $\log_{\left(\frac{1}{2}\right)} 20$ or $\log_{\left(\frac{1}{2}\right)} 50$

**e)** $\log(5+\sqrt{7})$ or $\log 5 + \log\sqrt{7}$ **f)** $(2\sqrt[3]{2})^{-6}$ or $2^{-11}$

**13.** Solve each of the following equations.

**a)** $\left(\frac{1}{4}\right)^{x-1} = 4^{2-3x}$ **b)** $2^x \times 5^x = 0.1 \times \left(10^{(x-1)}\right)^5$

**c)** $\left(\frac{1}{4}\right)^{3x} - \left(\frac{1}{8}\right)^{x-1} = 128$ **d)** $2^{2x+2} = 9(2^x) - 2$

**e)** $9^{x+1} + 3^{x+2} - 18 = 0$ **f)** $28 - 2\,log_2\sqrt{2} = 4 \times 3^{2x+5} - 3^{4x+8}$

**g)** $\frac{81^{5-2x} \times 243^{x-2}}{9^{5x-1}} = \frac{1}{3}$ **h)** $9^{1+\log_3 x} - 3^{1+\log_3 x} - 210 = 0$

**14.** State the universe and solve each of the following equations.

**a)** $\log_2(x+2) + \log_2(x-1) = 2$ **b)** $\log_3(x^2 - 8x) = 2$

**c)** $\frac{\log x}{\log(5x-3)} = 1$ **d)** $\frac{2+\log x}{3-\log x} = 5$

**e)** $\log(3x^2+1)-\log(3+x)=\log(3x-2)$ **f)** $\frac{\log(x^2+13)}{\log(x+5)}=2$

**g)** $\log(3x-1)-\log(3x+1)=\log 16$ **h)** $\log_3[1+\log_3(2^x-7)]=1$

**i)** $3\sqrt{\log x}+2\log\sqrt{x^{-1}}=2$ **j)** $\log_4(x+12)\cdot\log_x 2=1$

**15.** For the function given below, find the $x$-intercept, the $y$-intercept, the asymptot, the domain, the range and sketch its graph.

**a)** $f(x)=-2+2^x$ **b)** $h(x)=-2+2^{-x}$

**c)** $f(x)=\log_2(x+2)$ **d)** $g(x)=\log_{\left(\frac{1}{2}\right)}(x-1)$

**16.** The initial size of a culture of bacteria is 1,000. After an hour, the bacteria count is 8,000.

**a)** Find a function that models the population after $t$ hours.

**b)** Find the population after 1.5 hours.

**c)** When will the population reach 15,000?

**d)** Sketch the graph of the population function.

**17.** Suppose that Birr 10,000 is invested in a saving account paying 7% interest per year.

**a)** Write the formula for the amount in the account after $t$ years if interest is compounded monthly.

**b)** Find the amount in the account after 5 years if interest is compounded daily.

**c)** How long will it take for the amount in the account to grow to 25,000 if interest is compounded semiannually?

# UNIT 4

# TRIGONOMETRIC FUNCTIONS

## Unit Outcomes

By the end of this unit, you will be able to:

- Define the basic trigonometric functions.
- Sketch graphs of basic trigonometric functions.
- Define reciprocals of basic trigonometric functions.
- Identify trigonometric identities.
- Solve some examples of real-life problems involving trigonometric equations.
- Conceptualize theorems on special triangles.

## Unit Contents


- ✓ Angle of elevation
- ✓ Period
- ✓ Co-terminal angle
- ✓ Radian
- ✓ Complementary angles
- ✓ Reference angle
- ✓ Supplementary angles
- ✓ Negative angle
- ✓ Trigonometry
- ✓ Trigonometric function
- ✓ Pythagorean identity
- ✓ Unit circle
- ✓ Angle of depression
- ✓ Periodic function
- ✓ Degree
- ✓ Angle in standard position
- ✓ Quadrantal angle

## Introduction

The word ‘trigonometry’ is derived from the Greek word ‘trigon’ and ‘metron’ and it means ‘measuring the sides of a triangle’. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyors for mapping out the new lands and engineers for other purposes. Currently, trigonometry is used in many areas such as seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analyzing a musical tone and in many other areas.

In grade 9, you studied the trigonometric ratios of acute angles as the ratio of the sides of a right-angled triangle. In this unit, we will study the trigonometric identities and application of trigonometric ratios in solving the problems and generalize the concept of trigonometric ratios to trigonometric functions and study their properties.

## 4.1 Radian Measure of Angle: Conversion between radian and degree measures

### Activity 4.1

1. What is an angle?
2. Discuss the initial and terminal side of an angle.
3. What is a positive angle? What is a negative angle?

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the **initial side** and the final position of the ray after rotation is called the **terminal side** of the angle. The point of rotation is called the vertex.


(I) Positive Angle

(II) Negative Angle

Figure 4.1

If the direction of rotation is anticlockwise, the angle is said to be positive and if it is clockwise, then the angle is negative. The measure of an angle is the amount of rotation performed to get the terminal side. One complete revolution from the position of the initial side is indicated in the figure 4.2.


Figure 4.2

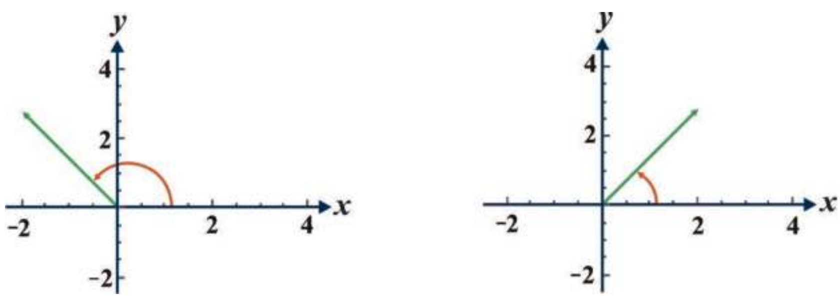
## Angles in standard position

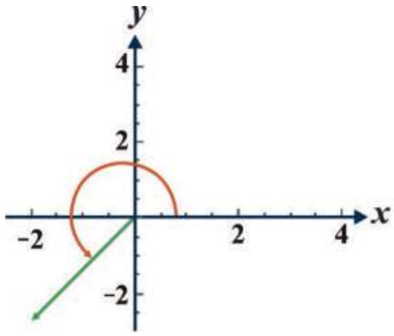
An angle in the coordinate system is said to be in standard position if

1. its vertex is at the origin.
2. its initial side lies on the positive $x$-axis

### Example 1

The following angles in figure4.3 are in standard position.





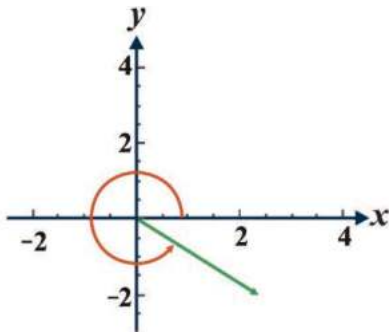




Figure 4.3

**Example 2**

The following are measures of different angles. Put the angles in standard position.

**a.** $200°$ **b.** $1125°$ **c.** $-900°$

**Solution:**

**a.** $200° = 180° + 20°$

**b.** $1125° = 3 \times 360° + 45°$

**c.** $-900° = 2 \times (-360°) + (-180°)$

We shall describe two units of measurement of an angle which are most commonly used **degree measure** and **radian measure.**

## 1. Degree measure

If a rotation from the initial side to terminal side is $(\frac{1}{360})^{th}$ of a complete revolution, the angle is said to have a measure of one degree, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute.

| | |
|---|---|
| $1°$ | $60'$ |
| $1'$ | $60''$ |

Some of the angles whose measures are $360°, 180°, 270°, 420°, -30°, -420°$ are shown in figure4.4.

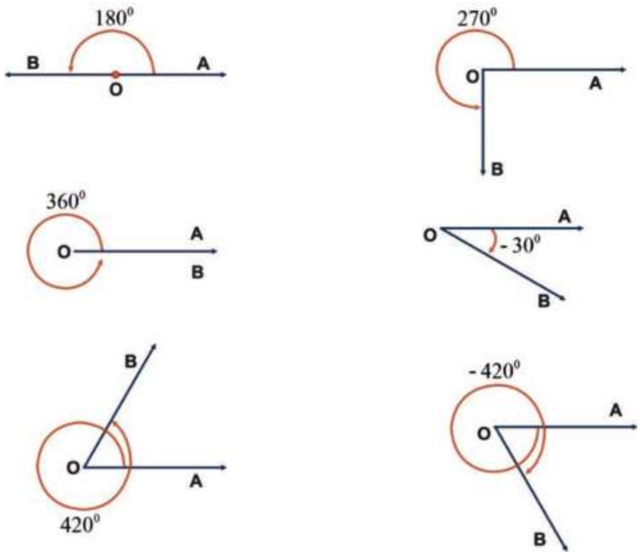



Figure 4.4

## Exercise 4.1

1. The following are measures of different angles. Put the angles in standard position.
   a. 765°  b. 245°  c. −740°
2. Draw the following angles. a. 270°  b. 90°  c. −270°

## 2. Radian measure

There is another unit for measurement of an angle called the radian measure. An angle at the center of a circle with radius $r$ which is subtended by an arc of length $r$ unit in a circle is said to have a measure of 1 radian. In the figure 4.5, $\overline{OA}$ is the initial side and $\overline{OB}$ is the terminal side of the central angle. The figures show the central angles whose measures are 1 and – 1 radians, respectively.

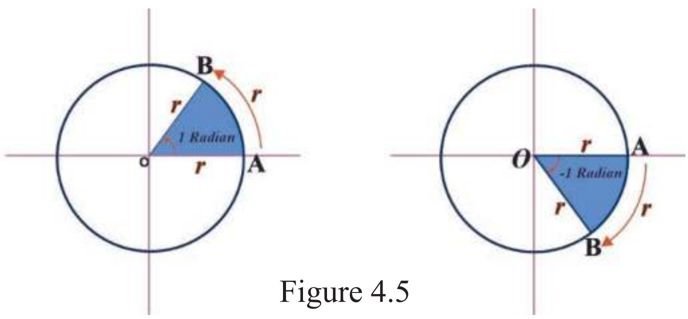



Figure 4.5

We know that the circumference of a circle of radius $r$ unit is $2\pi r$. Thus, one complete revolution of the initial side subtends an angle of $2\pi r$ radian. It is well-known that equal arcs of a circle subtend equal angle at the center. Since in a circle of radius $r$, an arc of length $r$ subtends an angle whose measure is 1 radian; an arc of length $l$ will subtend an angle whose measure is $\frac{l}{r}$ radian. Thus, if in a circle of radius $r$, an arc of length $l$ subtends an angle $\theta$ at the center, we have $\theta = \frac{l}{r}$ or $l = r\,\theta$.

## Relation between degree and radian measure

Since a circle subtends at the central angle whose radian measure is $2\pi$ and its degree measure is $360°$, it follows that $2\pi$ radian $=360°$ or π radian $= 180°$. The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. The relation between degree measures and radian measure of some common angles are given in table 4.1.

Table 4.1

| Degree | 30° | 45° | 60° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|---|---|
| Radian | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

**Note**

When an angle is expressed in radian, the word 'radian' is frequently omitted Thus, $\pi = 180°$ and $\frac{\pi}{4} = 45°$ are written with the understanding that $\pi$ and $\frac{\pi}{4}$ are radian measures.

Therefore, we can convert degrees to radians using the relations:

$$\text{Radian measure} = \frac{\pi}{180} \times \text{degree measure}$$

and radians to degrees using the relation

$$\text{Degree measure} = \frac{180}{\pi} \times \text{radian measure.}$$

### Example 1

Convert 120° into radian.

**Solution:**

We know $180° = \pi$.

$$120° = \frac{\pi}{180°} \times 120° = \frac{2}{3}\pi.$$

### Example 2

Convert $\frac{4}{3}\pi$ radian into degree measure.

**Solution:**

$$\frac{4}{3}\pi = \frac{180°}{\pi} \times \frac{4}{3}\pi = 240°. \text{ Hence, } \frac{4}{3}\pi \ is\ 240°.$$

### Example 3

Convert 6 radians into degrees.

**Solution:**

We know that $\pi$ radian $= 180°$.

$$\text{Hence, } 6 \text{ radians} = \frac{180}{\pi} \times 6 \ (\text{Use } \pi \approx \frac{22}{7})$$

$$= \frac{1080 \times 7}{22} = \frac{3780}{11} \text{ degree}$$

## Exercise 4.2

1. Convert each of the following degrees into radian:

   a. $30°$ b. $60°$ c. $240°$ d. $270°$ e. $-330°$ f. $220°$

2. Find the degree measures of angles which have the following radian measures:

   a. $\frac{\pi}{10}$ b. $\frac{5\pi}{4}$ c. $\frac{-3\pi}{5}$ d. $\frac{-\pi}{12}$ e. $\frac{11\pi}{15}$

3. Convert each of the following radians into degrees: (Use $\pi \approx \frac{22}{7}$)

   a. $\frac{110}{7}$ radian b. $\frac{44}{7}$ radian

### Example 4

Find the radius of the circle in which a central angle of 30° intercepts an arc of length 11 cm ( $\pi = \frac{22}{7}$ ).

**Solution:**

Here, an arc length $l = 11\text{cm}$, and $\theta = 30° = \frac{30\pi}{180} = \frac{\pi}{6}$.

Since, $r = \frac{l}{\theta}$, we have radius of the circle,

$$r = \frac{11\times6}{\pi} = \frac{11\times6\times7}{22} = 21 \text{ cm}.$$

## Example 5

The minute hand of a watch is 1.5  long. How long does its tip move in 40 minutes?

**Solution:**

In 60 minutes, the minute hand of a watch completes one revolution. Therefore,

in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution.

Therefore, $\theta = \frac{2}{3}\times 360°$ or $\frac{4\pi}{3}$.

Hence, the required distance travelled $l$ is calculated as follows:

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} = 2\pi \text{ units}.$$

## Example 6

If the arcs of the same lengths in two circles subtend angles 65° and 210° at the center, find the ratio of their radii.

**Solution:**

Let $r_1$ and $r_2$ be the radii of the two circles. It is given that

$$65° = \frac{\pi}{180}\times 65 = \frac{13\pi}{36} \text{ and } 210° = \frac{\pi}{180}\times 210 = \frac{7\pi}{6}.$$

Let l be the length of each of the arcs. Then, $l = \mathrm{r}_1\theta_1 = \mathrm{r}_2\theta_2$ which gives

$\frac{13\pi}{36}\times \mathrm{r}_1 = \frac{7\pi}{6}\times \mathrm{r}_2$, i.e., $13\pi\mathrm{r}_1 = 42\pi\mathrm{r}_2$ . Hence, $\mathrm{r}_1:\mathrm{r}_2 = 42:13$.

## Exercise 4.3

1. Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).
2. The minute hand of a watch is 1.5 long. How long does its tip move in 15 minutes?

# 4.2 Basic Trigonometric Functions

In grade 9, we studied trigonometric ratios for acute angles as the ratio of sides of a right-angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

## Activitv 4.2

1. Draw an isosceles triangle $ABC$ in which angle $C$ is a right angle and $AC = 2$ cm.
   a. What is $m(\angle A)$?
   b. Calculate the length of $\overline{AB}$.
   c. Find sin45°, cos45° and tan45°.
2. Discuss Right-angled Triangle and Pythagoras Theorem.

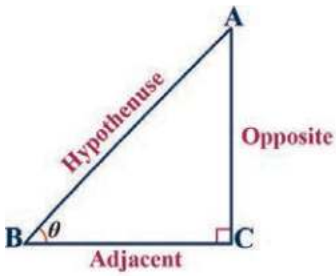



Figure (a)

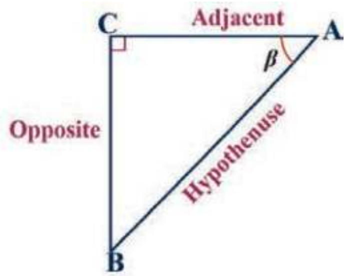



Figure (b)

Figure 4.6

In the figure 4.6, for a given right-angled triangle, the hypotenuse is the side which is opposite to the angle of right angle and it is the longest side of the triangle. For the angle marked by $\theta$ in figure 4.6(a), $\overline{AC}$ is the side opposite to angle $\theta$ and $\overline{BC}$ is the side adjacent to the angle $\theta$ .

Similarly, for the angle marked by $\beta$ in figure 4.6(b), $\overline{BC}$ is the side opposite to angle $\beta$ and $\overline{AC}$ is the side adjacent to the angle $\beta$.

## 4.2.1 The sine, cosine and tangent functions

Trigonometric functions are originally used to relate the angles of a triangle to the length of the sides of a triangle.

In this section, the same upper-case letter denotes a vertex of a triangle and the measure of the corresponding angle; the same lower-case letter denotes an edge of the triangle and its length. Given an acute angle $A = \alpha$ of a right-angled triangle, the hypotenuse $h$ (or $AB$) is the side that connects the two acute angles. The side $b$ $(or\ AC)$ adjacent to $\alpha$ is the side of the triangle that connects $\alpha$ to the right angle. The third side $a$ $(\ or\ BC)$ is said to be opposite to $\alpha$.

If the angle $\alpha$ is given, then all sides of the right-angled triangle are well-defined up to a scaling factor.

For any triangle $\Delta ABC$, with an angle $\alpha$, the sine, cosine and tangent functions will be defined as follows:

$$\sin\alpha = \frac{\text{opposite side to } \alpha}{\text{hypotenuse side to } \alpha} = \frac{BC}{AB} = \frac{a}{h}$$

$$\cos\alpha = \frac{\text{adjacent side to } \alpha}{\text{hypotenuse side to } \alpha} = \frac{AC}{AB} = \frac{b}{h} \text{ and}$$

$$\tan\alpha = \frac{\text{opposite side to } \alpha}{\text{adjacent side to } \alpha} = \frac{BC}{AC} = \frac{a}{b}.$$

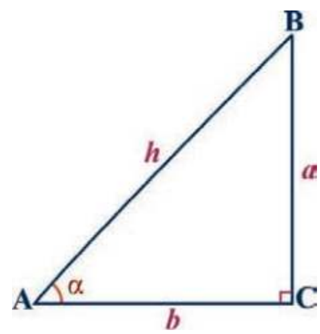



Figure 4.7

### Example 1

Find the values of the trigonometric ratios of angle $\theta$ in figure 4.8 where $P(5,12)$ is a point on the terminal side of $\theta$.

**Solution:**

Before we find the values of the trigonometric ratios, we need to find the length of the missing side length (hypotenuse). If $r$ is the length of hypotenuse, we use the Pythagoras Theorem as

$$r = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144}$$
$$= \sqrt{169} = 13.$$

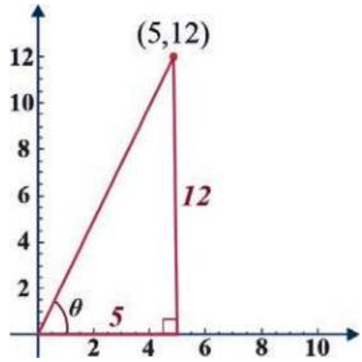



Figure 4.8

Now we can find the values of the three trigonometric ratios. Hence,

$\sin\theta = \frac{\text{opposite side to } \theta}{\text{hypotenuse}} = \frac{12}{13}$, $\cos\theta = \frac{\text{adjacent side to } \theta}{\text{hypotenuse}} = \frac{5}{13}$ and

$$\tan\theta = \frac{\text{opposite side to } \theta}{\text{adjacent side to } \theta} = \frac{12}{5}.$$

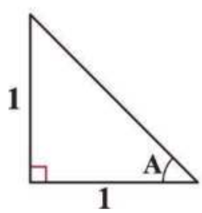
## Exercise 4.4

1. Evaluate the sine, cosine and tangent of angle $\theta$ if $\theta$ is in standard position and its terminal side contains the given point $(x, y)$.

   a. $(4,3)$ b. $(6, 8)$ c. $(1, \sqrt{2})$ d. $(\frac{\sqrt{2}}{2}, \frac{1}{2})$

2. Find the values of the trigonometric ratios of angle $A$ in figure 4.9.

a.




b.

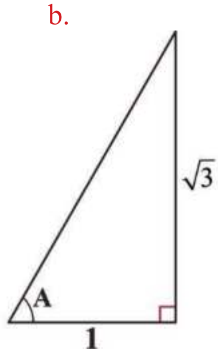



Figure 4.9

3. Almaz wants to find the value of $x$ in the given $\Delta$MNO, where $\cos\theta = \frac{1}{2}$. How you help Almaz?

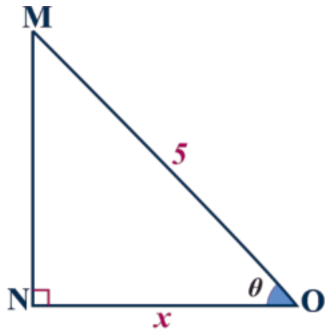



Figure 4.10

## Example 2

Calculate all angles and sides if the hypotenuse in a right-angled triangle $ABC$ is equal to 5.

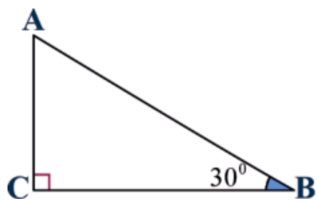



Figure 4.11

**Solution:**

When solving problems such as this, it is advisable to draw a sketch. In this example, either sine or cosine can be used. The order in which the problem is solved doesn't affect the final result. Angle at the point $B$ is equal to 30° and the opposite side of this angle is $AC$.

$$\sin 30° = \frac{AC}{AB}, \qquad AC = \sin 30° \times 5$$

$$AC = 2.5$$

Adjacent side is $BC$. So, cosine is used.

$$\cos 30° = \frac{BC}{AB}, \qquad BC = \cos 30° \times 5$$

$$BC = \frac{\sqrt{3}}{2} \times 5 = \frac{5\sqrt{3}}{2} \text{ and } m(\angle A) = 60°.$$

In mathematics, the trigonometric functions (also called circular functions or angle functions) are real functions which relate to an angle of a right-angled triangle to ratios of two side lengths.

## Exercise 4.5

1. Calculate all angles and sides if the hypotenuse of an isosceles right-angled triangle $ABC$ is equal to 4.

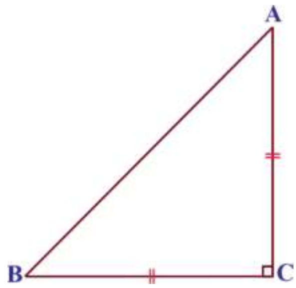



Figure 4.12

2. Ali wants to find the exact length of the shadow cast of a 15 m lamppost when the angle of elevation of the sun is 60°. What is the length of the shadow cast?

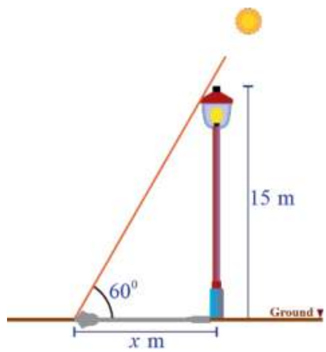



Figure 4.13

3. A kite in the air has a string tied to the ground as shown in figure 4.14. If the length of the string is 100 m, find the height of the kite above the ground when the string is taut and its inclination is 30° to the horizontal.

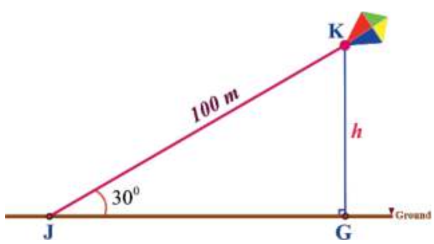



Figure 4.14

## The unit circles

### Activity 4.3

Based on the right-angled triangle labeled $A$ and $B$ as shown in figure 4.15.

a. Find cosine of angle A

b. Compute sine of angle A

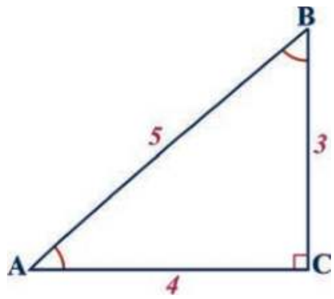



Figure 4.15

The **unit circle** is in the $xy$-plane. It is a circle with a radius of 1 and centered at the origin.

Now let us draw a right-angled triangle with the same acute angles a hypotenuse of 1 unit long. We find that the side opposite to angle A is $\frac{3}{5} = 0.6$ long and the side adjacent to angle A is $\frac{4}{5} = 0.8$ long.

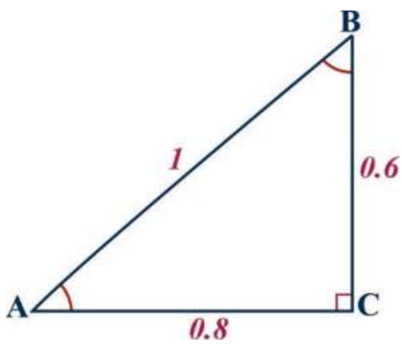
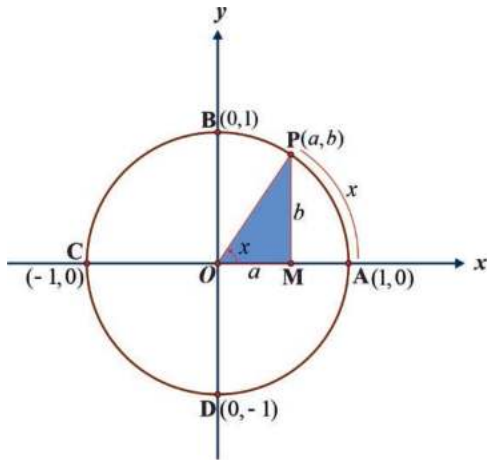



Figure 4.16

$\cos A = \frac{0.8}{1} = 0.8$ and $\sin A = \frac{0.6}{1} = 0.6$. Thus, the length of the side adjacent is numerically equal to the cosine of the angle, and the length of the side opposite is numerically equal to the sine of the angle. Because of this result, we can use a circle whose center is the origin and whose radius is 1 unit long to help us visualize the values of the cosine and sine of the central angle.

- Plot the points $(1,0),(0,1),(-1,0)$, and $(0,-1)$ on the $xy$- coordinate system.

Let $P\ (a,b)$ be any point on the circle with angle $AOP = x$ radian, i.e., length of arc $AP = x$ as shown figure4.17.




We define $\cos x = \frac{a}{1} = a$ and

$\sin x = \frac{b}{1} = b$.

So, the point $(a,b) = (\cos x, \sin x)$.

Since $\Delta$OMP is a right-angled triangle, we have $(\mathrm{OM})^2 + (\mathrm{MP})^2 = (\mathrm{OP})^2$ *or*

$a^2 + b^2 = 1$.

Thus, for any point on the unit circle,

we have $a^2 + b^2 = 1$, *or* $\cos^2 x + \sin^2 x = 1$.

Figure 4.17

Since one complete revolution subtends an angle of $2\pi$ radian at the center of the circle $\mathrm{m}(\angle\mathrm{AOB}) = \frac{\pi}{2}$, $\mathrm{m}(\angle\mathrm{AOC}) = \pi$ and $\mathrm{m}(\angle\mathrm{AOD}) = \frac{3\pi}{2}$.

All angles which are integral multiples of $\frac{\pi}{2}$ are called quadrantal angles. The coordinates of the points $A, B, C$ and $D$ are respectively $(1,0),(0,1),(-1,0)$ and $(0,-1)$.

Therefore, for quadrantal angles, we have

$$\cos 0 = 1 \qquad \sin 0 = 0$$

$$\cos\frac{\pi}{2} = 0 \qquad \sin\frac{\pi}{2} = 1$$

$$\cos\frac{3\pi}{2} = 0 \qquad \sin\frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point $P$, we again come back to the same point $P$. Thus, we also observe that if $x$ increases (or decreases) by any integral multiple of $2\pi$, the values of sine and cosine functions do not change.

Thus,

$$\sin(2n\pi + x) = \sin x, n \in Z$$

$$\cos(2n\pi + x) = \cos x, n \in Z.$$

Furthermore, $\sin x = 0$ , if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, ..$ i.e., when $x$ is an integral multiple of $\pi$.

$\cos x = 0$, if $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ..i.e.,$ when $x$ is an odd integral multiple of $\frac{\pi}{2}$.

Thus

$\sin x = 0$, implies $x = n\pi$, where $n$ is any integer,

$\cos x = 0$, implies $x = (2n + 1)\frac{\pi}{2}$, where $n$ is any integer.

In grade 9, we discussed the values of the trigonometric ratios for $0°, 30°, 45°, 60°$ and $90°$.

The values of trigonometric functions for these angles are the same as those of trigonometric ratios in table 4.2.

Table 4.2

| degree | 0° | 30° | 45° | 60° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|---|---|---|
| radian | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| $\sin x$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ | 0 |
| $\cos x$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $-1$ | 0 | 1 |
| $\tan x$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | Not Defined | 0 | Not Defined | 0 |

## Exercise 4.6

1. Using the unit circle, find the values of the sine, cosine and tangent functions of the following quadrantal angles:

   a. 0°  b. 450°  c. 540°  d. 630°

b. Fill in the blank with numbers.

| degree | $-360°$ | $-450°$ | $-270°$ | $-180°$ | $-90°$ | $90°$ | $180°$ | $720°$ |
|---|---|---|---|---|---|---|---|---|
| $\sin x$ | | | | | | | | |
| $\cos x$ | | | | | | | | |
| $\tan x$ | | | | | | | | |

## Sign of trigonometric functions

Let $P(a, b)$ be a point on the unit circle with center at the origin such that $m(\angle AOP) = x$. If $m(\angle AOQ) = -x$, the coordinate of the point $Q(a, -b)$(see Figure 4.18).

Therefore, $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.

Since for every point $P(a, b)$ on the unit circle where $-1 \leq a \leq 1$ and $-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$, for all $x$.

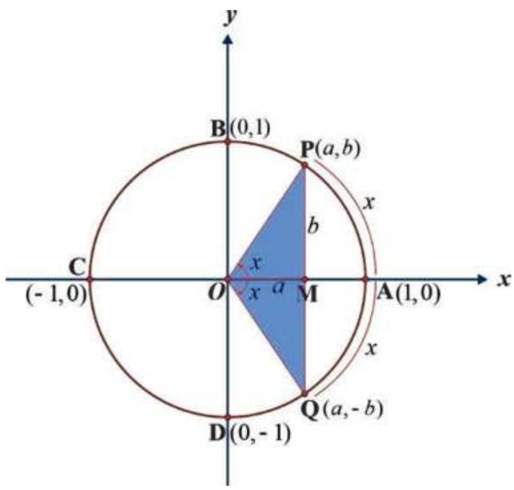



Figure 4.18

We have learnt that in the first quadrant $(0 < x < \frac{\pi}{2})$, $a$ and $b$ are both positive; in the second quadrant $(\frac{\pi}{2} < x < \pi)$, $a$ is negative and $b$ positive; in the third quadrant $\pi < x < \frac{3\pi}{2}$, $a$ and $b$ are both negative and in the fourth quadrant $(\frac{3\pi}{2} < x < 2\pi)$, $a$ is positive and $b$ is negative.

Therefore, sin$x$ is positive for $0 < x < \pi$ and negative for $\pi < x < 2\pi$. Similarly, cos$x$ is positive for $0 < x < \frac{\pi}{2}$, and negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants.

In fact, we have the following table.

Table 4.3

| $y =$ | Quadrants | | | |
|---|---|---|---|---|
| | I | II | III | IV |
| sin$x$ | + | + | - | - |
| cos$x$ | + | - | - | + |
| tan$x$ | + | - | + | - |

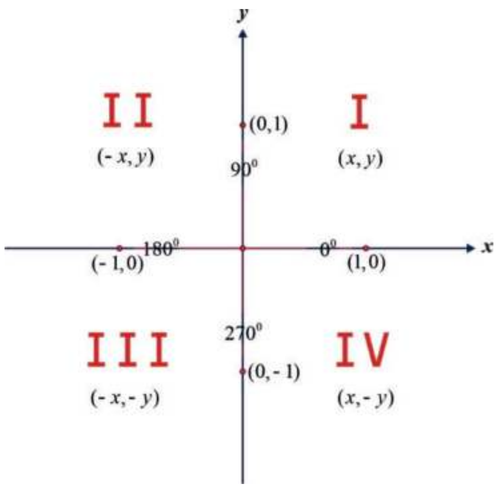



Figure 4.19

### Example 1

Find the quadrant where angle $x$ makes $\sin x > 0$ and $\cos x < 0$.

**Solution:**

Here, $\sin x > 0$ in the first and second quadrants, and $\cos x < 0$ in the second and third quadrants. Thus, when $\sin x > 0$ and $\cos x < 0$ are both satisfied, $x$ is the angle of the second quadrant.

## Exercise 4.7

1. Find the quadrant where angle $x$ is located for the following conditions.
   a. $\sin x < 0$ and $\cos x > 0$
   b. $\sin x > 0$ and $\tan x < 0$
   c. $\cos x > 0$ and $\tan x < 0$
2. Fill in the blank with numbers.

| Degree | 0° | 120° | 135° | 150° | 210° | 240° | 330° |
|---|---|---|---|---|---|---|---|
| radian | 0 | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{7\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{11}{6}\pi$ |
| $\sin x$ | | | | | | | |
| $\cos x$ | | | | | | | |
| $\tan x$ | | | | | | | |

## Reciprocal trigonometric functions

**Note**

We can define other trigonometric functions in terms of sine and cosine:

It is convenient to have a name for the reciprocal of the sine, cosine, and tangent of a given angle $\theta$. We call these reciprocal functions the secant(sec), cosecant(csc), and cotangent(cot) and define them as follows:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}.$$

### Example 1

Evaluate sec30°

**Solution:**

We know sec30° is the reciprocal of cos 30°. Therefore, we have

$$\cos 30° = \frac{\sqrt{3}}{2}.$$

$$\text{So, } \sec 30° = \frac{1}{\cos 30^0} = \frac{2}{\sqrt{3}}.$$

### Example 2

Evaluate $\sec 60° + \csc 30°$.

**Solution:**

Since $\cos 60° = \frac{1}{2}$ and $\sec 60°$ is the reciprocal of $\cos 60°$ we obtain

$$\sec 60^0 = \frac{1}{\cos 60°} = \frac{1}{\frac{1}{2}} = 2.$$

Since $\sin 30° = \frac{1}{2}$ and $\sin 30°$ is the reciprocal of $\csc 30°$ we have

$$\csc 30° = \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2.$$

Thus, $\sec 60° + \csc 30° = 2 + 2 = 4$.

## Exercise 4.8

1. Evaluate the following

   a. $\sec 45°$ b. $\sec\frac{2\pi}{3}$ c. $\sec\left(-\frac{\pi}{6}\right)$

   d. $\csc 30°$ e. $\csc\frac{3\pi}{4}$ f. $\csc(-300°)$

   g. $\cot 60°$ h. $\cot\frac{5\pi}{6}$ i. $\cot\left(-\frac{5\pi}{4}\right)$

2. Evaluate the following trigonometric expressions.

   a) $\sec\frac{10\pi}{3} + \csc\left(-\frac{7\pi}{2}\right)$

   b) $\sec 330° + \cot 480°$

## 4.2.2 Trigonometric values of angles

Trigonometry angles are the angles given by the ratios of the trigonometric functions. Trigonometry deals with the study of the relationship between angles and the sides of a triangle. The angle value ranges from 0° to 360°, . The important angles in trigonometry are $0°, 30°, 45°, 60°, 90°, 180°, 270°$ and $360°$. And the important six trigonometric ratios or functions are sine, cosine, tangent, cosecant, secant and cotangent.

## Complementary angles

### Activity 4.4

Based on the isosceles right-angled triangle $ABC$ in Figure 4.20, calculate the length of the hypotenuse $AB$ and verify that the two angles $A$ and $B$ are congruent.

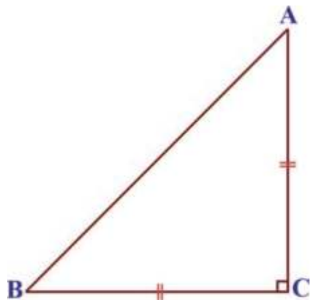



Figure 4.20

Two angles are said to be **complementary** angles if they are added up to 90°. In the figure 4.21,

$60° + 30° = 90°$. Hence, these two angles are complementary. Each angle of the complementary angles is called the "complement" of the other angle. Here, 60° is the complement of 30°. Similarly, 30° is the complement of 60°. Thus, the complement of an angle is calculated by subtracting the angle from 90°. So, the complementary angle of angle $x°$ is $(90 - x)°$.

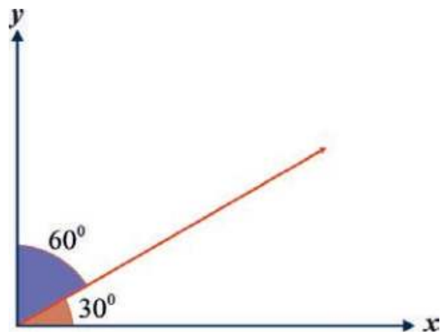



Figure 4.21

### Example 1

Find the complementary angle of 57°.

**Solution:**

The complementary angle of 57° is obtained by subtracting it from 90°, that is,

$$90° - 57° = 33°.$$

Thus, the complementary angle of 57° is 33°.

### Example 2

Evaluate the following angles:

a. 54°  b. 30°

**Solution:**

$\sin 54° = \sin(90° - 36°) = \cos 36°$, and $\sin 30° = \sin(90° - 30°) = \cos 60°$.

**Note**

Note the following on acute angle $\alpha$ and its complementary angle $(\pi/2 - \alpha)$.

- $\sin(\pi/2 - \alpha) = \cos\alpha$
- $\cos(\pi/2 - \alpha) = \sin\alpha$
- $\tan(\pi/2 - \alpha) = \cot\alpha$
- $\cot(\pi/2 - \alpha) = \tan\alpha$

### Example 3

If $\sin 3A = \cos(A - 26°)$, where $3A$ is an acute angle, find the value of $A$.

**Solution:**

Given that, $\sin 3A = \cos(A - 26°)$ ….(1)

Since, $\sin 3A = \cos(90° - 3A)$, we can write (1) as:

$$\cos(90° - 3A) = \cos(A - 26°)$$

$$90° - 3A = A - 26°$$

$90° + 26° = 3A + A$ which implies $4A = 116°$.

So, $A = \frac{116}{4} = 29°$.

Therefore, the value of $A$ is 29°.

## Exercise 4.9

Answer each of the following questions:

1. Find the numerical value of
   a. sin 30° and cos 60°  b. sin 45° and cos 45°
   c. What can you generalize or deduce based on your answer to $a$ and b above?

2. If $\sin 31° = 0.515$, then what is $\cos 59°$?
3. If $\sin\theta = \frac{3}{5}$, then what is $\cos(90° - \theta)$?
4. If $\cos\alpha = \frac{4}{5}$ , then what is $\sin(\frac{\pi}{2} - \alpha)$?
5. If $\sin\theta = k$ , then what is $\cos(\frac{\pi}{2} - \theta)$
6. If $\tan\beta = \frac{m}{n}$, then what is $\frac{1}{\tan(90° - \beta)}$
7. If $\cos(4a) = \sin(a - 20°)$ where $4a$ is an acute angle. Find the value of $a$.

## Reference angle $(\theta_R)$

The reference angle of any angle always lies between 0° and 90°. It is the angle between the terminal side of the angle and the $x$- axis. The reference angle depends on the quadrant's terminal side.

The steps to find the reference angle of an angle depend on the quadrant of the terminal side:

- We first determine its coterminal angle which lies between 0° and 360°.
- We then see the quadrant of the coterminal angle.
- If the terminal side is in the first quadrant ( 0° to 90°), then the reference angle is the same as our given angle.

For example, if the given angle is 25°, then its reference angle $(\theta_R)$ is also 25° as shown in Figure 4.22.

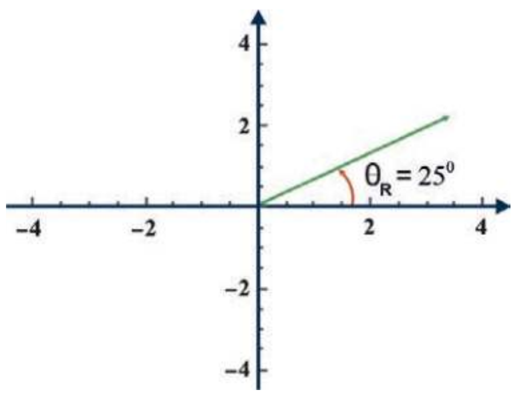



Figure 4.22

- If the terminal side is in the second quadrant ( 90° to 180°), then the reference angle $(\theta_R)$ is **180° minus the given angle.**

For example, if the given angle is 100°, then its reference angle is 180° – 100° = 80° as shown in figure 4.23.

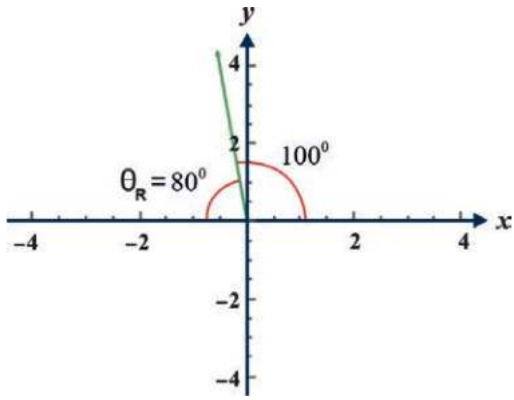



Figure 4.23

- If the terminal side is in the third quadrant (180° to 270°), then the reference angle $(\theta_R)$ is the **given angle minus 180°**.

For example, if the given angle is 215°, then its reference angle is 215° – 180° = 35° as shown in figure 4.24.

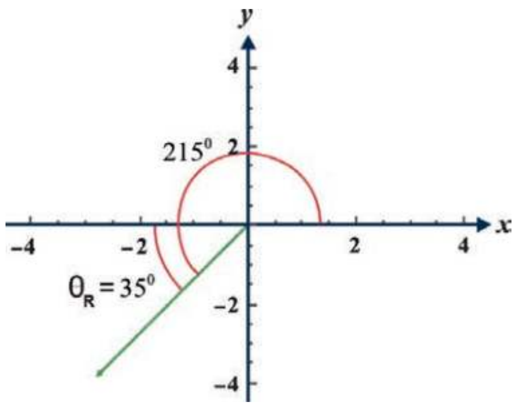



Figure 4.24

- If the terminal side is in the fourth quadrant (270° to 360°), then the reference angle $(\theta_R)$ is **360° minus the given angle.**

For example, if the given angle is 330°, then its reference angle is 360° – 330° = 30° as shown in figure 4.25.

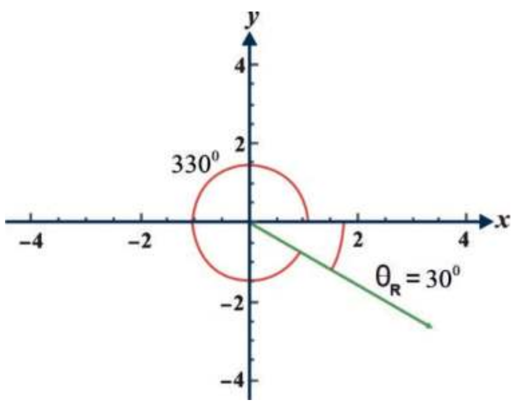



Figure 4.25

## Example 1

Find the reference angle $\theta_R$ for the angles

**a.** $\theta = 129°$ **b.** $\theta = 245°$ **c.** $\theta = 320°$

**Solution:**

**a.** Since $\theta = 129°$ is a second quadrant angle,

$$\theta_R = 180° - 129° = 51°.$$

**b.** Since $\theta = 245°$ is a third quadrant angle,

$$\theta_R = 245° - 180° = 65°.$$

**c.** Since $\theta = 320°$ is a fourth quadrant angle,

$$\theta_R = 360° - 320° = 40°.$$

**Note**

The value of the trigonometric function of a given angle θ and the values of the corresponding trigonometric functions of the reference angle $\theta_R$ are the same in absolute value.

## Example 2

Express the sine, cosine and tangent of $155°$ in terms of its reference angle.

**Solution:**

Remember that an angle with measure $155°$is a second quadrant angle. In the second quadrant, only sine is positive. So, $\theta_R = 180° - 155° = 25^0$.

Therefore, $\sin 155°, = \sin 25°$, $\cos 155° = -\cos 25°$ and $\tan 155° = -\tan 25°$.

## Exercise 4.10

1. Find the reference angle $\theta_R$ for the angles

a. $\theta = 109°$ b. $\theta = 345°$ c. $\theta = 190°$ d. $\theta = 140°$

e. $\theta = \frac{5\pi}{3}$ f. $\theta = \frac{7\pi}{4}$ g. $\theta = \frac{4\pi}{3}$

2. Express the sine, cosine and tangent of $150°$ in terms of its reference angle.

## Supplementary angles

The supplementary angles are angles that exist in pairs summing up to $180°$. So, supplementary angle of an angle $x$ is $180°$ $minus$ $x$.

### Example 1

Determine whether the following pairs of angles are supplementary or not.

**a.** $50°$ and $130°$ **b.** $70°$ and $100°$.

**Solution:**

We know that two angles are supplementary if their sum is $180°$.

**a.** $50° + 130° = 180°$

Since the sum is $180°$, the given angles are supplementary.

**b.** $70° + 100° = 170°$

Since the sum is not$180°$, the given angles are **not** supplementary.

## Co-terminal angles

### Activity 4.5

What are co-terminal angles?

Co-terminal angles are angles that have the same initial side and share the terminal sides. The co-terminal angles occupy the standard position, though their values are different. They are on the same sides, in the same quadrant and their vertices are identical. When the angles are moved clockwise or anticlockwise, the terminal sides coincide at the same angle.

Consider $45°$. Its standard position is in the first quadrant because its terminal side is also in the first quadrant. Look at the image as shown in figure 4.26. On full rotation anticlockwise, $45°$ reaches its terminal side again at $405°$. So, $405°$ coincides with

45° in the first quadrant. On full rotation clockwise, 45° reaches its terminal side again at $-315°$. $-315°$ coincides with 45° in the first quadrant.

Thus 405° and $-315°$ are co-terminal angles of 45°.

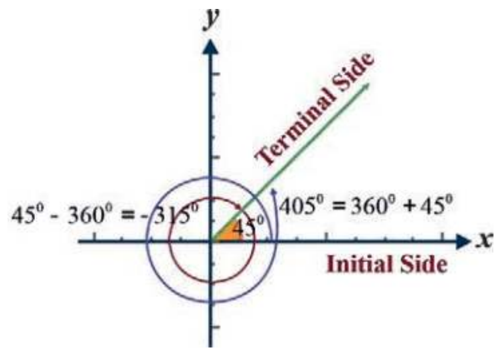



Figure 4.26

The formula to find the co-terminal angles of an angle $\theta$ depends upon whether it is in terms of degrees or radians:

1. Degrees: $\theta \pm 360n$, where n is an integer.
2. Radian: $\theta \pm 2n\pi$, where n is an integer.

So, $45°, -315°, 405°, -675°, 765°$ ..... are all co-terminal angles. They differ only by a number of complete circles. We can conclude that two angles are said to be co-terminal if the difference between the angles is a multiple of 360° (or $2\pi$ if the angle is in terms of radians).

## Example 1

Find two co-terminal angles of 30°.

**Solution:**

The given angle is $\theta = 30°$. The formula to find the co-terminal angle is

$\theta \pm 360°n$ Then find the first co-terminal angle using $n = 1$.

The corresponding co-terminal angle $= \theta + 360°n$

$$= 30° + 360°(1)$$
$$= 390°.$$

To find the second co-terminal angle when $n = -2$ (clockwise).

Then, the corresponding co-terminal angle $= \theta + 360°n$

$$= 30° + 360°(-2)$$
$$= -690°$$

**Note**

From the above explanation, we can find the co-terminal angle(s) of any angle either by adding or subtracting multiples of 360° (or $2\pi$) from the given angle. So, we actually do not need to use the co-terminal angles formula to find the co-terminal angles. Instead, we can either add or subtract multiples of 360° (or $2\pi$) from the given angle to find its co-terminal angles.

## Example 2

Find a co-terminal angle of $\frac{\pi}{4}$.

**Solution:**

The given angle is $\theta = \frac{\pi}{4}$, which is in radians.

So, we add or subtract multiples of $2\pi$ from $\frac{\pi}{4}$ to find its co-terminal angles.

Let us subtract $2\pi$ from the given angle as

$$\frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}.$$

Thus, one of the co-terminal angles of $\frac{\pi}{4}$ is $-\frac{7\pi}{4}$.

**What are positive and negative co-terminal angles?**

Co-terminal angles can be positive or negative. In one of the above examples, we found that 390° and − 690° are co-terminal angles of 30°.

Here, 390° is the positive co-terminal angle of 30° and −690° is the negative co-terminal angle of 30°.

**Note**

$\theta \pm 360\, n$, where $n$ takes a positive value when the rotation is anticlockwise and takes a negative value when the rotation is clockwise. So, we decide whether to add or subtract multiples of 360° (or $2\pi$) to get positive or negative co-terminal angles, respectively.

## Exercise 4.11

1. Determine whether the following pair of angles are supplementary or not.

   a. $90°$ and $100°$ b. $135°$ and $45°$ c. $\frac{\pi}{6}$ and $\frac{2\pi}{3}$

2. Find a positive and a negative angle which are co-terminal with angle $55°$.
3. Find a positive and a negative angle which are co-terminal with angle $\frac{\pi}{3}$.

### Example 1

Find a reference angle $\theta_R$ of $495°$.

**Solution:**

First let us find the co-terminal angle of $495°$. The co-terminal angle is

$495° - 360° = 135°$.

The terminal side lies in the second quadrant.

Thus, the reference angle is $180° - 135° = 45°$.

Therefore, the reference angle of $495°$ is $45°$.

### Example 2

Evaluate: $\sin 780°$

**Solution:**

$780° = 720° + 60° \quad = 2 \times 360° + 60°$

Therefore, $\sin 780° = \sin 60° = \frac{\sqrt{3}}{2}$.

## Exercise 4.12

Evaluate the following expressions.

a. $\sin 390°$ b. $\cos\frac{10\pi}{3}$ c. $\tan(-420°)$

d. $\sin(-660°)$ e. $\cos\frac{41\pi}{4}$ f. $\tan\left(-\frac{19\pi}{3}\right)$

g. $4\cos 135°$ h. $\frac{5}{3}\cos 300°$ i. $-2\cos(-150°)$

## Graphs of the sine, cosine and tangent functions

To sketch the trigonometry graphs of the functions of sine, cosine and tangent, we need to know the period, phase, and amplitude, maximum and minimum turning points. The graphical representation of sine, cosine and tangent functions are explained here briefly. Students can learn how to graph a trigonometric function here along with activities based on it.

Sine, cosine and tangent are the three important trigonometric ratios based on which functions are defined. In these trigonometry graphs, we use $x$-axis for values of the angles in radians and the $y$-axis values of the function at each given angle.

## The graph of the sine function

### Activity 4.6

1. Complete the following table of values for the function $y = \sin\theta$

Table 4.4

| $\Theta$ | $-360°$ | $-270°$ | $-90°$ | $-30°$ | $0°$ | $30°$ | $90°$ | $270°$ | $360°$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

2. Sketch the graph of $y = \sin\theta$ using table 4.4.
3. What is the period of sine function?

### Example 1

Draw the graph of $y = \sin\theta$.

**Solution:**

To determine the graph of $y = \sin\theta$, we construct a table of values for $y = \sin\theta$ where $-360° \leq \theta \leq 360°$.

The table below shows some of the values of $y = \sin\theta$ in the given interval. To draw the graph, you mark the values of $\theta$ on the horizontal axis and the value of $y$ on the vertical axis. Then you plot the points and connect them using smooth curve (see figure 4.27)

Table 4.5

| $\theta$ in degree | $-360°$ | $-330°$ | $-270°$ | $-240°$ | $-210°$ | $-180°$ | $-150°$ | $-120°$ | $-90°$ | $-90°$ | $-45°$ | $-30°$ | $0°$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ in radian | $-2\pi$ | $\frac{-11\pi}{6}$ | $-\frac{3\pi}{2}$ | $-\frac{4\pi}{3}$ | $-\frac{7\pi}{6}$ | $-\pi$ | $-\frac{5\pi}{6}$ | $-\frac{2\pi}{3}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{3}$ | $-\frac{\pi}{4}$ | $-\frac{\pi}{6}$ | 0 |
| $y = \sin\theta$ | 0 | 0.5 | 1 | 0.87 | 0.5 | 0 | −0.5 | −0.87 | −1 | −0.87 | −0.71 | −0.5 | 0 |

| $\theta$in degree | $30°$ | $60°$ | $60°$ | $90°$ | $120°$ | $150°$ | $180°$ | $210°$ | $240°$ | $270°$ | $300°$ | $330°$ | $360°$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ in radian | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{11\pi}{6}$ | $2\pi$ |
| $y = \sin\theta$ | 0.5 | 0.71 | 0.87 | 1 | 0.87 | 0.5 | 0 | −0.5 | −0.87 | −1 | −0.87 | −0.5 | 0 |

After a complete revolution (every $360°$ *or* $2\pi$), the values of the sine function repeat themselves. This means

$\sin 0° = \sin(0° \pm 360°) = \sin(0° \pm 2 \times 360°) = \sin(0° \pm 3 \times 360°)$, etc

$\sin 90° = \sin(90° \pm 360°) = \sin(90° \pm 2 \times 360°) = \sin(90° \pm 3 \times 360°)$, etc

$\sin 180° = \sin(180° \pm 360°) = \sin(180° \pm 2 \times 360°) = \sin(180° \pm 3 \times 360°)$,etc

In general, $\sin\theta° = \sin(\theta^0 \pm 360°) = \sin(\theta° \pm 2 \times 360°) = \sin(\theta° \pm 3 \times 360°)$,etc

A function that repeats its value at regular intervals is called a periodic function. The sine function repeats after every $360°$ ($or\ 2\pi$). Therefore, $360°$ ($or\ 2\pi$) is called the period of the sine function.

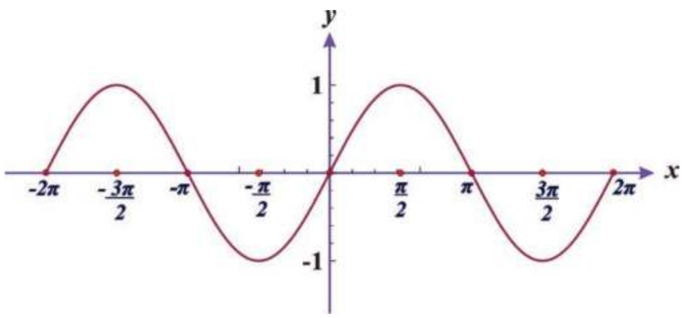



Figure 4.27 The graph of $y = \sin\theta$ for $-2\pi \leq \theta \leq 2\pi$

## Exercise 4.13

1. Draw graph of the following functions.

   $y = 2\sin x$, for $-2\pi \leq x \leq 2\pi$

2. Find the values A to H in the following graph of $y = \sin x$.

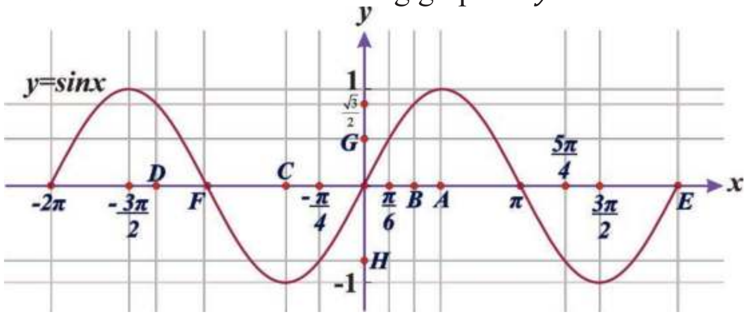



Figure 4.28

## Domain and range:

For any angle $\theta$ taken on the unit circle, there is some point $P(x, y)$ on its terminal side. Since $y = \sin\theta$ , the function $y = \sin\theta$ is defined for every angle $\theta$ taken on the unit circle.

Therefore, the domain of the sine function is the set of all real numbers.

**Note**

The domain of the sine function is $\{\theta: \theta \in \mathbb{R}\}$. The range of the sine function is $\{y(\theta): -1 \leq y \leq 1, \theta \in \mathbb{R}\}$.

## The graph of the cosine function

**Activity 4.7**

1. Complete following table of values for the function $y = \cos\theta$.

Table 4.6

| $\theta$ | $-360°$ | $-270°$ | $-90°$ | $-30°$ | $0°$ | $30°$ | $90°$ | $270°$ | $360°$ |
|---|---|---|---|---|---|---|---|---|---|
| $y = \cos\theta$ | | | | | | | | | |

2. Sketch the graph of $y = \cos\theta$ using table 4.6.
3. What is the period of cosine function?

From the above activity, you can see that $y = \cos\theta$ is never less than $-1$ or more than $+1$ (see figure4.29). Just like the sine function, the cosine function is periodic at every $360°$ (or $2\pi$) radians. Therefore,$360°$ (or $2\pi$) is called the period of the cosine function.

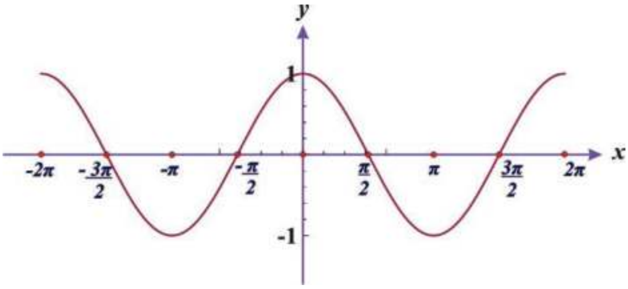



Figure 4.29 Graph of $y = \cos\theta$

The domain of the cosine function is the set of all real numbers. The range of the cosine function is $\{y: -1 \leq y \leq 1\}$.

## Exercise 4.14

1. Draw graph of the following functions.

$$y = \cos x, \text{ for } -2\pi \leq x \leq 2\pi.$$

2. Find the values A to H in the following graph of $y = \cos x$.

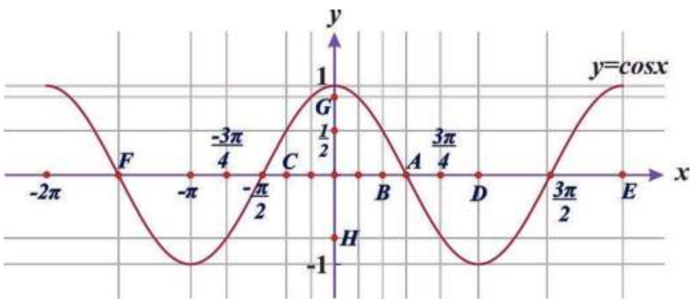



Figure 4.30

## Graph of the tangent function

### Activity 4.8

1. Complete the following table of values for the function $y = \tan\theta$.

Table 4.7

| $\theta$ | $-360$ | $-270$ | $-90°$ | $-45°$ | $-30°$ | $0°$ | $30°$ | $45°$ | $90°$ | $270°$ | $360°$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = \tan\theta$ | | | | | | | | | | | |

2. Sketch the graph of the function $y = \tan\theta$ using table 4.7.
3. What is the period of tangent function?
4. For which values of $\theta$, $y = \tan\theta$ is not defined?

### Example 2

Draw graph of the function $y = \tan\theta$, where $-360° \leq \theta \leq 360°$.

Table 4.8 shows some of the values of $y = \tan\theta$ in the given interval.

Table 4.8

| $\theta$ | $-360°$ | $-315°$ | $-270°$ | $-225°$ | $-180°$ | $-135°$ | $-90°$ | $-45°$ | $0°$ |
|---|---|---|---|---|---|---|---|---|---|
| $\theta$ in radian | $-2\pi$ | $-\frac{7\pi}{4}$ | $-\frac{3\pi}{2}$ | $-\frac{5\pi}{4}$ | $-\pi$ | $-\frac{3\pi}{4}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | 0 |
| $y = \tan\theta$ | 0 | 1 | undefine | $-1$ | 0 | 1 | undefine | $-1$ | 0 |

| $\theta$ | $30°$ | $45°$ | $90°$ | $135°$ | $180°$ | $225°$ | $270°$ | $315°$ | $360°$ |
|---|---|---|---|---|---|---|---|---|---|
| $\theta$ in radian | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
| $y = \tan\theta$ | 0.56 | 1 | undefined | $-1$ | 0 | 1 | undefine | $-1$ | 0 |

Look at the tangent function $y = \tan\theta$ graph shown in figure 4.31.

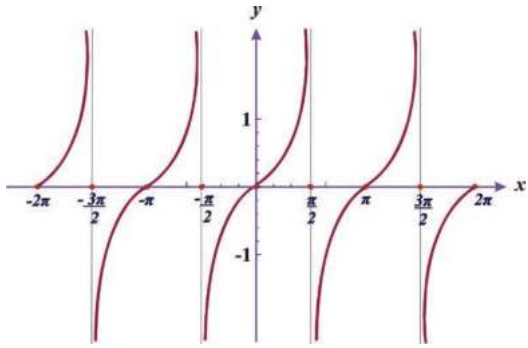



Figure 4.31 The graph of $y = \tan\theta$ for $-2\pi \leq \theta \leq 2\pi$

Do you notice that the pattern of curves is repeating after an interval of $\pi$?

Also, observe that the values of $\tan x$ increases as $x$ increases in $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi)$, $(\pi, \frac{3\pi}{2})$, ….

The domain of the tangent function is $\{\theta : \theta \neq \frac{n\pi}{2}$, where $n$ is an odd integer$\}$.

The range of the tangent function is the set of all real numbers.

From the graph we see that the tangent function repeats itself every $180°$ or $\pi$ radians.

Therefore, the period of the tangent function is $180°$ or $\pi$ radians.

## Exercise 4.15

1. Draw graph of the function $y = \tan x$, for $-2\pi \leq x \leq 2\pi$.
2. The following is the graph of $y = \tan x$.
   a. Find the values $y$ of the following points A, B, C and D.
   b. Find the values of $x$ of the following points E, F and G.

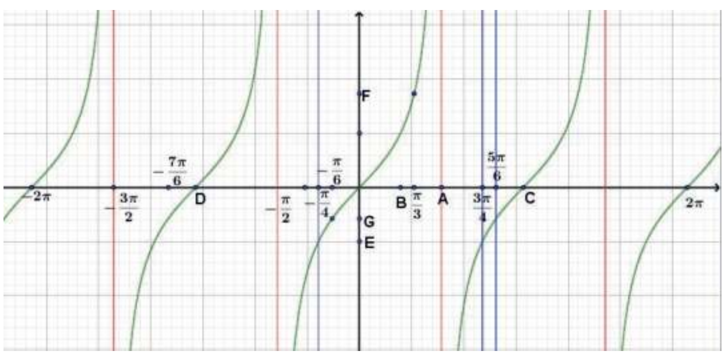



Figure 4.32

# 4.3 Trigonometric Identities and Equations

By considering a right-angled triangle, the trigonometric identities or equations are formed using trigonometry ratios for all the angles. Using trigonometry identities, we can express each trigonometric ratio in terms of other trigonometric ratios. If any of the trigonometry ratio value is known to us, then we can find the values of other trigonometric ratios. We can also solve trigonometric identities, using these identities as well.

## Trigonometric Identities

There are basically three trigonometric identities, which we learn in this topic. They are:

1. $\cos^2\theta + \sin^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

Here, we will prove the above trigonometric identities. Take an example of a right-angled triangle $\Delta ABC$ as shown in figure 4.33.

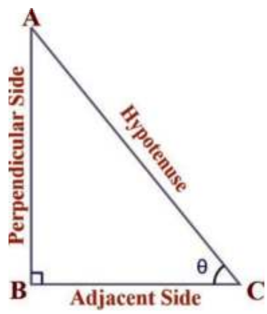



Figure 4.33

**Proof of Trigonometric Identities**

In a right-angled triangle, by the Pythagoras Theorem, we know

$$(\text{Perpendicular side length})^2 + (\text{Base length})^2 = (\text{Hypotenuse length})^2$$

Therefore, in $\Delta ABC$, we have;

$$(AB)^2 + (BC)^2 = (AC)^2 \qquad \ldots(1)$$

Dividing equation (1) by $(AC)^2$ we get,

$$\frac{(AB)^2}{(AC)^2} + \frac{(BC)^2}{(AC)^2} = 1$$

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1 \quad \text{by rule of exponent}$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1 \quad \left(\text{Since } \sin\theta = \frac{AB}{AC} \text{ and } \cos\theta = \frac{BC}{AC}\right)$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \ldots(2)$$

For all angles equation (2) is satisfied.

### Example 1

$x$ is the angle of the third quadrant. When $\sin x = -\frac{3}{5}$, find the value of $\cos x$.

**Solution:**

$\sin^2 x + \cos^2 x = 1$ and $\cos^2 x = 1 - \sin^2 x = 1 - \left(-\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$. But $x$ is the angle of the third quadrant, so $\cos x < 0$.

Thus $\cos x = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$.

## Exercise 4.16

a. If $x$ is the angle in the second quadrant and $\cos x = -\frac{\sqrt{5}}{3}$, find the value of $\sin x$.

b. If $x$ is the angle in the fourth quadrant and $\sin x = -\frac{1}{3}$, find the value of $\cos x$.

Again, when we divide equation (1) by $(AB)^2$, we get

$$\frac{(AB)^2}{(AB)^2} + \frac{(BC)^2}{(AB)^2} = \frac{(AC)^2}{(AB)^2}$$

$$1 + \frac{(BC)^2}{(AB)^2} = \frac{(AC)^2}{(AB)^2}$$

$$1 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2 \quad \text{(By rule of exponent)}$$

$$1 + (\cot\theta)^2 = (\csc\theta)^2 \quad \left(\text{Since } \cot\theta = \frac{BC}{AB} \text{ and } \csc\theta = \frac{AC}{AB}\right)$$

$$1 + \cot^2\theta = \csc^2\theta \quad \ldots(3)$$

Therefore, it proves that for all values of $\theta$, equation (3) is satisfied.

Let's see what we get if we divide equation (1) by $(BC)^2$, we get

$$\frac{(AB)^2}{(BC)^2} + 1 = \frac{(AC)^2}{(BC)^2}$$

$$\left(\frac{AB}{BC}\right)^2 + 1 = \left(\frac{AC}{BC}\right)^2 \quad \text{by rule of exponent}$$

$(\tan\theta)^2 + 1 = (\sec\theta)^2$ (Since $\tan\theta = \frac{AB}{BC}$ and $\sec\theta = \frac{AC}{BC}$)

$1 + \tan^2\theta = \sec^2\theta$ ...(4)

### Example 2

$x$ is the angle of the fourth quadrant. When $\tan x = -2$, find the values of $\sec x$ and $\cos x$.

**Solution:**

$\sec^2 x = 1 + \tan^2 x = 1 + (-2)^2 = 5$

$x$ is the angle of the fourth quadrant and $\sec x = \frac{1}{\cos x}$.

In the fourth quadrant $\cos x > 0$, so $\sec x > 0$.

Therefore $\sec x = \sqrt{5}$ and $\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{5}}$.

## Exercise 4.17

Answer the following questions.

a. $x$ is the angle of the third quadrant. If $\tan x = 3$, find the values of $\sec x$ and $\cos x$.

b. $x$ is the angle of the second quadrant. If $\tan x = 3$, find the values of $\sec x$ and $\cos x$

## Addition and subtraction of identities

The formulas for the addition and subtraction theorems of sine and cosine are expressed as in the following:

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta,$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta,$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta,$$

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta.$$

### Example 1

Find the values of trigonometric expressions:

**a.** sin75° **b.** cos75°

**Solution:**

**a.** $\sin 75° = \sin(30° + 45°)$

$= \sin 30° \cos 45° + \cos 30° \sin 45°$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$

**b.** $\cos 75° = \cos(30° + 45°)$

$= \cos 30° \cos 45° - \sin 30° \sin 45°$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$

## Exercise 4.18

Find the values of the following trigonometric expressions.

a. sin 105° b. cos 105° c. sin 15° d. cos 15° e. $\sin\frac{\pi}{12}$ f. $\cos\frac{\pi}{12}$

## Double angle identities

Formulas expressing trigonometric functions of an angle $2\theta$ in terms of an angle $\theta$:

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

### Example 1

If $x$ is an angle of the second quadrant and $\sin x = \frac{3}{5}$, find the following values.

**a.** $\cos x$ **b.** $\sin 2x$ **c.** $\cos 2x$

**Solution:**

**a.** $\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$x$ is the angle of the second quadrant, so $\cos x < 0$

Thus $\cos x = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$

**b.** $\sin 2x = 2\sin x\cos x = 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right) = -\frac{24}{25}$

**c.** $\cos 2x = \cos^2 x - \sin^2 x = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

## Half angle identities

The half-angle identities for the sine and cosine are derived from two of the cosine identities described earlier.

$\cos 2\theta = 2\cos^2\theta - 1$. Let $\theta = \frac{\alpha}{2}$, then $\cos\left(2\times\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$$

$$2\cos^2\frac{\alpha}{2} = \cos\alpha + 1$$

$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$. So, $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$.

Similarly, $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$ and $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$.

The sign of the two preceding functions depends on the quadrant in which the resulting angle is located.

### Example 1

Find the exact value for cos 15° using the half-angle identity.

**Solution:**

$$\cos 15^\circ = \cos\frac{30^\circ}{2} = \pm\sqrt{\frac{1+\cos 30^\circ}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$= \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$. (15° is in the first quadrant, hence $\cos 15^\circ > 0$)

## Exercise 4.19

1. Assume $x$ is the angle in the first quadrant. When $\cos x = \frac{2}{3}$, find the following values.

   a. $\sin x$  b. $\sin 2x$  c. $\cos 2x$

2. Find the exact values using the half - angle identity.

   a. $\sin 15°$  b. $\sin\frac{\pi}{8}$  c. $\cos\frac{3\pi}{8}$

## Trigonometric equations

Trigonometric equations can be solved using the algebraic methods, trigonometric identities and values.

### Example 1

Solve the equation $2\cos\theta - 1 = 0, \quad 0 \leq \theta < 2\pi$.

**Solution:**

When we rearrange the above equation, we get $\cos\theta = \frac{1}{2}$.

For the reference angle $\theta_R$, we have $\cos\theta_R = \frac{1}{2}$ and hence, $\theta_R = \frac{\pi}{3}$. Using the reference angle $\theta_R$, we determine the solution $\theta$ in the interval $[0, 2\pi]$ of the given equation $\cos\theta = \frac{1}{2}$ suggests that $\cos\theta$ is positive and that means the terminal side of $\theta$ solution to the given equation is either in quadrant I or IV. Hence,

$\theta_1 = \theta_R = \frac{\pi}{3}$ and $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

### Example 2

Solve the trigonometric equation $2\sin\theta = -1$.

**Solution:**

Rewrite the above equation in simple form as shown below:

$\sin\theta = -\frac{1}{2}$ The reference angle $\theta_R$ such that $\sin\theta_R = \frac{1}{2}$ is $\theta_R = \frac{\pi}{6}$.

Use the reference angle $\theta_R$ to determine the solutions $\theta_1$ and $\theta_2$ on the interval $[0\,,2\pi)$ of the given equation. The equation $\sin\theta = -\frac{1}{2}$ suggests that $\sin\theta$ is negative and that means the terminal side of angle $\theta$ is either in quadrant III or IV as shown in the unit circle figure 4.34.

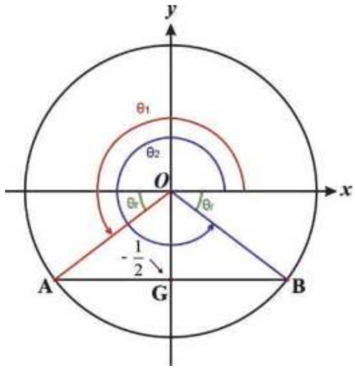



Figure 4.34

Hence,

$$\theta_1 = \pi + \theta_R \quad \text{or} \quad \theta_2 = 2\pi - \theta_R$$

$$= \pi + \frac{\pi}{6} \qquad\qquad = 2\pi - \frac{\pi}{6}$$

$$= \frac{7\pi}{6} \qquad\qquad = \frac{11\pi}{6}$$

Use the solutions on the interval $[0\,,2\pi)$ to find all solutions by adding multiples of $2\pi$ as follows:

$\theta_1 = \frac{7\pi}{6} + 2n\pi$ and $\theta_2 = \frac{11\pi}{6} + 2n\pi$, where $n$ is an integer.

## Exercise 4.20

Solve the following trigonometric equations.

a. $\sin\theta = \frac{1}{\sqrt{2}}$ where $0° \leq \theta < 2\pi$.

b. $\sqrt{2}\cos x = -1$ when $0° \leq \theta < 360°$

c. $2\sin x + \sqrt{3} = 0$ when $0° \leq \theta < 360°$

d. $2\cos x = \sqrt{3}$

e. $\sec x - \sqrt{2} = 0$

## Example 3

Find all the solutions of the trigonometric equation $\sqrt{3}\,\sec\theta + 2 = 0$

**Solution:**

Using the identity $\sec\theta = \frac{1}{\cos\theta}$, we rewrite the equation in the form

$$\cos\theta = -\frac{\sqrt{3}}{2}.$$

Find the reference angle $\theta_R$ by solving $\cos\theta_R = -\frac{\sqrt{3}}{2}$ for $\theta_R$ acute. Accordingly, $\theta_R = \frac{\pi}{6}$. Using the reference angle $\theta_R$ in the interval $[0, 2\pi)$ of the given equation $\cos\theta = -\frac{\sqrt{3}}{2}$ . This suggests that $\cos\theta$ is negative and that means the terminal side of angle $\theta$ is either in quadrants II or III as shown in figure 4.35.

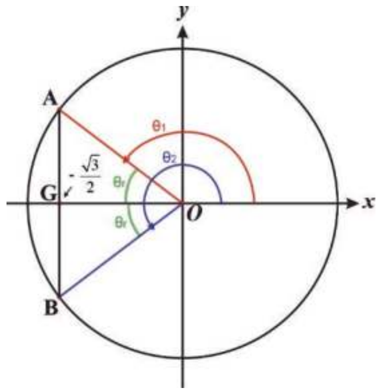



Figure 4.35

Hence,

$$\theta_1 = \pi - \theta_R \quad \text{or} \quad \theta_2 = \pi + \theta_R$$
$$= \pi - \frac{\pi}{6} \qquad\qquad = \pi + \frac{\pi}{6}$$
$$= \frac{5\pi}{6} \qquad\qquad = \frac{7\pi}{6}$$

Use the solutions on the interval $[0, 2\pi)$ to find all solutions by adding multiples of $2\pi$ as follows:

$\theta_1 = \frac{5\pi}{6} + 2n\pi$ and $\theta_2 = \frac{7\pi}{6} + 2n\pi$ where $n$ is an integer.

## Exercise 4.21

Solve the following trigonometric equations.

a. $2 + \sqrt{3}\csc\theta = 0$ for $0° \le \theta < 2\pi$ .

b. $3\sqrt{2} + 3\csc\theta = 0$ for $0° \le \theta < 2\pi$ .

## 4.4 Applications of Trigonometric Functions

### Activity 4.9

1. Assume that a skateboard ramp at a park has an inclination of 45° and its base is 12 m long as shown in figure 4.36. So, what is the length of the ramp?

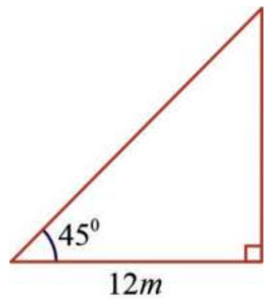



Figure 4.36

2. You are 50 m away from a river. Rather than walking directly to the river, you walk 100 m along a straight path to the river's edge as shown in figure 4.37. What is the angle between this path and the river's edge?

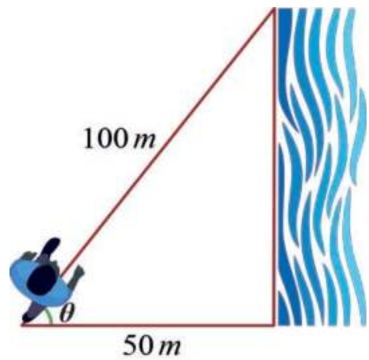



Figure 4.37

In the figure 4.38, angle labeled 1 indicates the angles of elevation. It is the angle by which the ground observer's line of vision must be raised or elevated with respect to the horizontal to see an object at B. The angle labeled 2 is the angle of depression. It is the angle by which an observer's line of vision at B must be lower or depressed with respect to the horizontal to see an object at *A*.

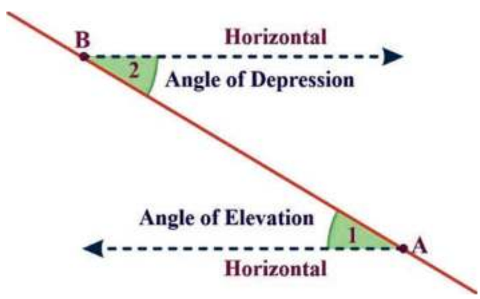



Figure 4.38

### Example 1

From the top of a vertical cliff 40 m high, the angle of depression of an object that is at the level of the base of the cliff is 60° as shown in figure 4.39. How far is the object from the base of the cliff (A)?

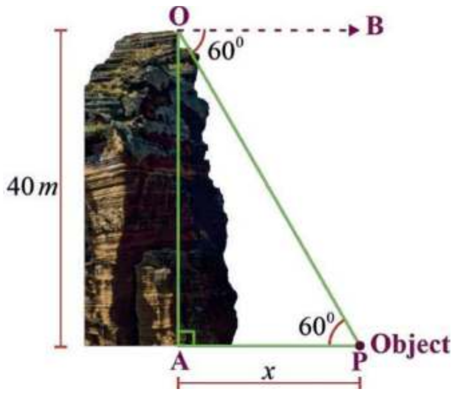



Figure 4.39

**Solution:**

Let $x$ be the distance of the object in meters from the base of the cliff (A). The angle of depression is 60°. Here, $m(\angle APO) = m(\angle BOP)$ because they are alternate angles.

$$\therefore m(\angle APO) = 60°$$

From triangle $APO$, we have:

$$\tan 60° = \frac{AO}{AP} = \frac{40}{x} \text{ which implies } x = \frac{40}{\tan 60°} = \frac{40}{\sqrt{3}}.$$

Therefore, the object is $\frac{40}{\sqrt{3}}$ m far from the cliff (A).

## Exercise 4.22

From the top of a vertical tree 10 $m$ high, the angle of depression of an object that is on the ground is 45° as shown in figure 4.40. How far is the object from the base of the tree?

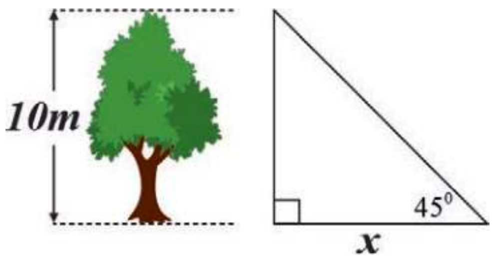



Figure 4.40

## Example 2

A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole if the angle is made by the rope with the ground level is 30° (see figure 4.41).

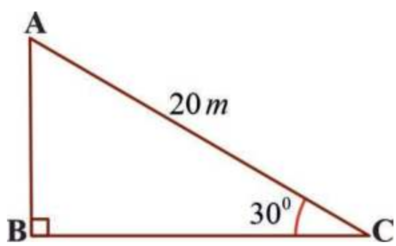



Figure 4.41

### Solution:

In right angle $\Delta ABC$,

$$\sin 30° = \frac{AB}{AC}$$

$$\frac{1}{2} = \frac{AB}{20} \implies AB = \frac{1}{2} \times 20 = 10$$

Therefore, the height of the pole is 10 m.

### Example 3

A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the original height of the tree before it breaks (see figure 4.42).

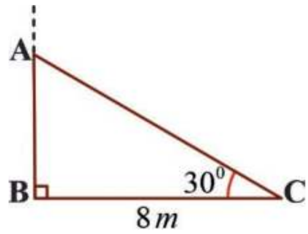


Figure 4.42

### Solution:

In right angle $\Delta ABC$,

$$\tan 30° = \frac{AB}{BC} \qquad \cos 30° = \frac{BC}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{8} \qquad \frac{\sqrt{3}}{2} = \frac{8}{AC}$$

$$AB = \frac{8}{\sqrt{3}} \text{ m} \qquad AC = \frac{16}{\sqrt{3}} \text{ m.}$$

Therefore, the height of the tree is $AB + AC = 8\sqrt{3}$ m.

## Exercise 4.23

Dana is standing on the ground and looking at the top of the tower with an angle of elevation of 30°. If he is standing 15 m away from the foot of the tower, can you determine the height of the tower?

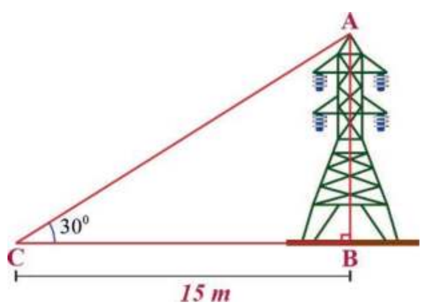


Figure 4.43

# Summary

1. An angle is determined by the rotation of a ray about its vertex from an initial position to a terminal position.
2. An angle is positive for anticlockwise rotation and negative for clockwise rotation.
3. An angle in the coordinate plane is in standard position if its vertex is at the origin and its initial side is along the positive $x$-axis.
4. Radian measure of angles: $2\pi = 360°$.
5. To convert degrees to radians, multiply by $\frac{\pi}{180°}$.
6. To convert radians to degree, multiply by $\frac{180°}{\pi}$.
7. If $\theta$ is an angle in standard position and $P(x, y)$ is a point on the terminal side of $\theta$, other than the origin $O(0,0)$ and r is the distance of point P from the origin O, then

   $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$

   $\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$ and $\cot\theta = \frac{x}{y}$

   $r = \sqrt{x^2 + y^2}$ (Pythagoras Theorem)

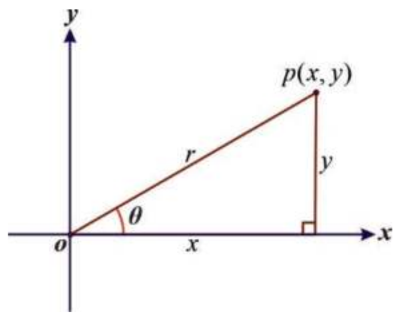



Figure 4.44

8. If $\theta$ is an angle in standard position, then

   $\sin(-\theta) = -\sin(\theta)$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$
9. Two angles are said to be complementary if their sum is equal to 90°. If $\alpha$ and $\beta$ are any two complementary angles, then

   $\sin\alpha = \cos\beta$ $\cos\alpha = \sin\beta$ $\tan\alpha = \frac{1}{\tan\beta}$
10. Any trigonometric function of an acute angle is equal to the coterminal of its complementary angles. That is, if $0° \le \theta \le 90°$, then

    $\sin\theta = \cos(90° - \theta)$ $\cos\theta = \sin(90° - \theta)$ $\tan\theta = \cot(90° - \theta)$

$\csc\theta = \sec(90° - \theta)$ $\sec\theta = \csc(90° - \theta)$ $\cot\theta = \tan(90° - \theta)$

**11.** If $\theta$ is an angle in standard position whose terminal side does not lie on either coordinate axis, then the reference angle $\theta_R$ for $\theta$ is the positive acute angle formed by the terminal side of $\theta$ and the $x$ −axis.

**12.** Two angles are said to be coterminal if the difference between the angles is a multiple of 360° (or $2\pi$ if the angle is in terms of radians).

**13.** Supplementary angles are angles that exist in pairs summing up to 180°. So, supplement of an angle $\theta$ is $(180° - \theta)$.

$\sin\theta = \sin(180° - \theta)$, $\cos\theta = -\cos(180° - \theta)$, $\tan\theta = -\tan(180° - \theta)$

**14.** Coterminal angles are angles in standard position (angles with the initial side on the positive $x$-axis) that have a common terminal side.

**15.** Coterminal angles have the same trigonometric values.

**16.** The domain of the sine function is the set of all real numbers.

**17.** The range of the sine function is $\{y: -1 \leq y \leq 1\}$.

**18.** The graph of the sine function repeats itself every 360° or $2\pi$.

**19.** The domain of the cosine function is the set of all real numbers.

**20.** The range of the cosine function is $\{y: -1 \leq y \leq 1\}$.

**21.** The graph of the cosine function repeats itself every 360° or $2\pi$.

**22.** The domain of the tangent function $is$ $\{\theta: \theta \neq n\frac{\pi}{2}, where\ n$ is an odd integer$\}$

**23.** The range of the tangent function is the set of all real numbers.

**24.** Trigonometric identities:

$\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $\cot^2 + 1 = \csc^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

## Review Exercise

**1.** Find the radian measure of the angles which have the following degree measures:

**a.** 135° **b.** 540° **c.** 210° **d.** 150°

**2.** Find the degree measure of the angles which have the following radian measures:

**a.** $\frac{\pi}{9}$ **b.** $2\frac{\pi}{3}$ **c.** $\frac{3\pi}{7}$ **d.** $\frac{20\pi}{9}$

**3.** What is the radian measure of each of the following angles?

**a.** $\frac{1}{2}$revolution anti-clockwise **b.** 5 revolution clockwise **c.** 330°

**d.** −225° **e.** 540° **f.** −360°

**4.** Convert 43.1025° to degree, minute and second form.

**5.** Use the table given at the end of the book to find the approximate value of:

**a.** sin40° **b.** tan40°

**6.** Find angle $A$ if:

**a.** $\sec A = 1.642$ **b.** $\sin A = 0.5831$

**7.** Use trigonometric table, reference angles, trigonometric functions of negative angles and periodicity of the functions to calculate the value of each of the following:

**a.** sin236 **b.** cos693°

**8.** Convert each of the following degrees to radians:

**a.** 225° **b.** 315° **c.** 330°

**d.** 420° **e.** 900° **f.** − 240°

**9.** Find two co-terminal angles for each of the following angles:

**a.** 65° **b.** 230° **c.** 790°

**d.** −674° **e.** −1545° **f.** 2060°

**10.** Convert each of the following angles in radians to degrees:

**a.** $\frac{9\pi}{14}$ **b.** $\frac{-7\pi}{15}$ **c.** $\frac{97\pi}{4}$ **d.** $7\pi$

**11.** Use a unit circle to find the values of sine, cosine, and tangent of $A$ when $A$ is:

**a.** $810°$ **b.** $-450°$ **c.** $-1080°$

**d.** $630°$ **e.** $900°$

**12.** Evaluate the sine, cosine and tangent of angle $\theta$ if $\theta$ is in standard position and its terminal side contains the given point $P(x, y)$:

**a.** $P(6, 8)$ **b.** $P(-6, 8)$ **c.** $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **d.** $P(-1, \sqrt{2})$

**13.** Find the values of sine, cosine, and tangent functions of A when A is in radian:

**a.** $\frac{3\pi}{4}$ **b.** $\frac{3\pi}{2}$ **c.** $\frac{-7\pi}{4}$

**d.** $\frac{-7\pi}{2}$ **e.** $\frac{-5\pi}{6}$

**14.** Find a reference angle for each of the following angles;

**a.** $130°$ **b.** $1030°$ **c.** $340°$

**d.** $-236°$ **e.** $-720°$

**15.** If $A$ is an acute angle, then find angle $A$ when:

**a.** $\cos 30° = \frac{1}{\sec A}$ **b.** $\sin A = \cos A$ **c.** $1 = \frac{2\cos A}{\sqrt{2}}$

**16.** If $A$ is an obtuse angle and $\sin A = \frac{4}{5}$, then evaluate

**a.** $cosA$ **b.** $tanA$ **c.** $cscA$ **d.** $secA$

**17.** Find the height of the tree if the angle of the elevation of its top changes from $25°$ to $50°$ and the observer advances 15 meters towards its base.

**18.** The angle of depression of the top and the foot of a flagpole as seen from the top of a building 145 meters away are $26°$ and $34°$, respectively. Find the heights of the pole and the building.

**19.** If $\cos A = \frac{5}{13}$ and $0° \leq A \leq 180°$, find the values of

**a.** $sinA$ **b.** $tanA$

**20.** Find the solution that satisfy $2 + \sqrt{3}\sec\theta = 0$ if $0° \leq \theta < 360°$.

**21.** Solve $3\sqrt{2} + 3\csc\theta = 0$ , if $0° \leq \theta < 2\pi$.

**22.** Evaluate $\frac{2}{3}\csc\frac{3\pi}{4} - \sec\frac{7\pi}{4}$.

**23.** Show that $\frac{\sec^2\theta - \tan^2\theta}{1+\cot^2\theta} = \sin^2\theta$.

**24.** Show that $\frac{\cos A}{1+\sin A} + \frac{1+\sin A}{\cos A} = 2\sec A$.

**25.** Evaluate $4\sin\frac{\pi}{4} + \sin\left(-\frac{\pi}{3}\right)$

**26.** Find the values of the six trigonometric functions of angle $a$ in the right-angled triangle shown in figure 4.45.

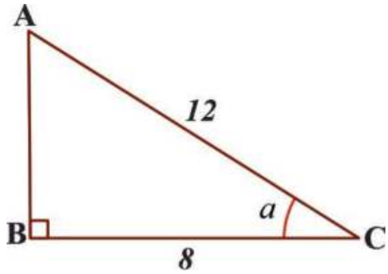



Figure 4.45

**27.** Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the right-angled triangle as shown in figure 4.46.

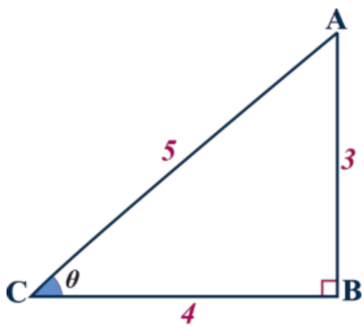



Figure 4.46

**28.** If $\theta$ is an angle in standard position and $P(6, 8)$ is a point in the terminal side of $\theta$, then evaluate the $\sec\theta$, $\csc\theta$, and $\cot\theta$.

**29.** If $\theta = 45°$, the find $\csc\theta$, $\sec\theta$ and $\cot\theta$.

**30.** Using unit circle, find the values of the cosecant, secant and cotangent functions if $\theta = -45°, -225°, -315°, 225°$, and $315°$.

**31.** Suppose the following points lie on the terminal side of an angle $\theta$. Find the secant, cosecant and cotangent functions of angle $\theta$.

**a.** $P(12,5)$ **b.** $P(-8,5)$ **c.** $P(2,0)$

**d.** $P\left(\frac{4}{5},-\frac{3}{5}\right)$ **e.** $P(\sqrt{2},\sqrt{5})$

**32.** If $\cot\theta = \frac{3}{8}$ and $\theta$ is in the first quadrant, find the other five trigonometric functions.

**33.** A cable tied to an electric pole is affixed at a point on the ground $x$ meters away from the foot of the pole to keep it upright. If the cable makes an angle $\theta$ with the ground, find is the length of the cable?

**34.** A man observed a pole of height 60 ft. According to his measurement, the pole cast a 20 ft. long shadow. Find the angle of elevation of the sun from the tip of the shadow using trigonometry.

# UNIT 5

# CIRCLE

## Unit Outcomes

By the end of this unit, you will be able to:

- **Explain the symmetrical properties of circles.**
- **Use the symmetrical properties of circles to solve related problems.**
- **Write angle properties of circles in their own words.**
- **Apply angle properties of circles to solve related problems.**
- **Find perimeters and areas of segments and sectors.**

## Unit Contents


- ✓ circumference
- ✓ sector
- ✓ symmetry
- ✓ segment
- ✓ perimeter
- ✓ quadrilateral
- ✓ circle
- ✓ ratio
- ✓ triangle
- ✓ arc
- ✓ area
- ✓ angle

# Introduction

You have learnt several concepts and principles in your lower grades. In the present unit you will learn more about circles. Perimeter and area of segment and a sector of a circle are the major topics covered in this unit.

## 5.1 Symmetrical Properties of Circles

### Activity 5.1

1. What is a circle?
2. Draw a circle and indicate its center, radius and diameter.
3. What is line of symmetry?
4. How many lines of symmetry does an equilateral triangle have?

A circle is the locus of points (set of points) in a plane each of which is equidistant from a fixed point in the plane. The fixed point is called the center of the circle and the constant distance is called its radius. Thus, the circle is defined by its center O and radius r.

A circle is also defined by two of its properties such as area and perimeter. Recall that area of a circle, $A = \pi r^2$ and perimeter of the circle, $P = 2\pi r$.

Observe that in a symmetrical figure the length of any line segment or the size of any angle in one half of the figure is equal to the length of the corresponding line segment or the size of the corresponding angle in the other half of the figure.

If in figure 5.1, point $P$ coincides with point $Q$ when the figure is about line $AB$ and if $\overline{PQ}$ intersects line $AB$ at $N$, then $\angle PNA$ coincides with $\angle QNA$ and therefore each is a right angle with $\overline{PN} \equiv \overline{QN}$. If $P$ and $Q$ are the corresponding points for a line of symmetry $AB$, the perpendicular bisector of $\overline{PQ}$ is $\overline{AB}$. Conversely, if $\overline{AB}$ is the perpendicular bisector of $\overline{PQ}$, then $P$ and $Q$ are corresponding points for the line of symmetry $AB$ and we say that $P$ is the image of $Q$ and $Q$ is the image of $P$ in line segment $AB$.

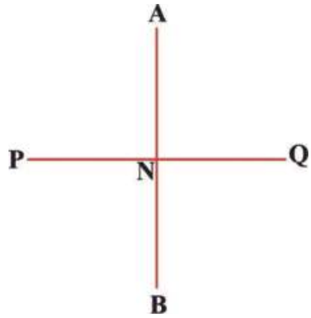


Figure 5.1

If we fold a circle over any of its diameters, then the parts of the circle on each side of the diameter will match up and the parts of the circle on each side of the diameter must have the same area. Thus, any diameter of a circle can be considered as a line of symmetry for the circle.
An object can have zero lines of symmetry or it can have infinite lines of symmetry.

## Example 1

Determine the number of lines of symmetry for the figure 5.2.

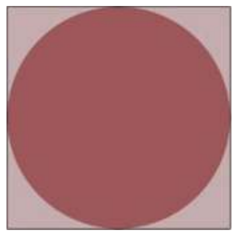


Figure 5.2

**Solution:**

We know that a circle has infinite lines of symmetry but as per the given Figure 5.2, a circle has been inscribed in a square. A square has 4 lines of symmetry. Therefore, the given figure 5.2 has 4 lines of symmetry as shown in figure 5.3.

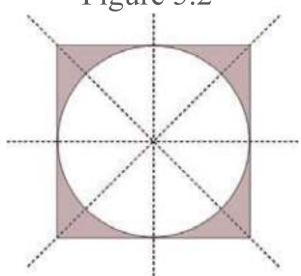


Figure 5.3

## Theorem 5.1

The line segment joining the center of a circle to the midpoint of a chord is perpendicular to the chord.

## Proof:

Given: A circle with center $O$ and a chord $\overline{PQ}$ whose midpoint is $M$ (see figure 5.4).

We want to prove that $\angle OMP$ is a right angle.

Draw the diameter $\overline{ST}$ through point M. Then, the circle is symmetric about $\overline{ST}$ and $\overline{PM} \equiv \overline{QM}$. So, $\overline{ST}$ is perpendicular bisector of $\overline{PQ}$ and hence $\angle OMP$ is a right angle.

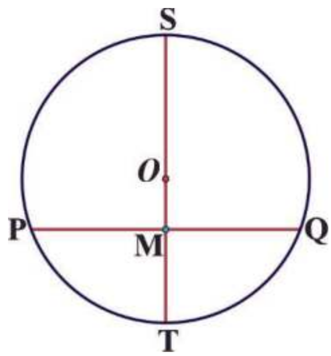



Figure 5.4

## Exercise 5.1

1. $\Delta ABC$ is an equilateral triangle and circle O is its circumcircle. How many lines of symmetry does figure 5.5 have?

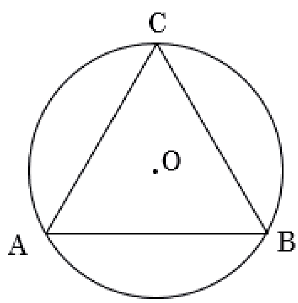



Figure 5.5

2. In figure 5.4 above, show that $\Delta OPM \equiv \Delta OQM$ .

## Characteristics of Chord (1)

### Theorem 5.2

The line segment drawn from the center of a circle perpendicular to a chord bisects the chord.

### Proof:

Given: A circle with center $O$ and $\overline{ON}$ is drawn from center $O$ perpendicular to the chord $AB$ as shown in figure 5.6.

We want to prove that $\overline{AN} \equiv \overline{NB}$.

Join $\overline{OA}$ and $\overline{OB}$.

1. $\overline{OA} \equiv \overline{OB}$ — radii of circle
2. $m(\angle ANO) = m(\angle BNO)$ — both equal to $90°$
3. $\overline{ON} \equiv \overline{ON}$ — common side
4. $\Delta AON \equiv \Delta BON$ — by RHS-criteria of congruency
5. $\overline{AN} \equiv \overline{BN}$ — by step 4

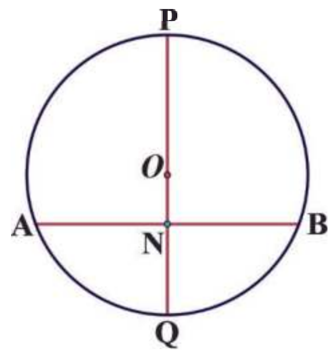



Figure 5.6

### Theorem 5.3

Equal chords of a circle are equidistant from the center of the circle.

### Proof:

Given: Chords $AB$ and $CD$ are equal in length.

Construction: Join points $A$ and $C$ with center $O$ and drop perpendiculars from $O$ to the chords $AB$ and $CD$ (see figure 5.7).

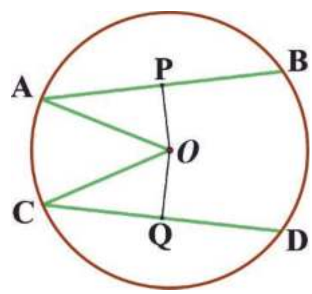



Figure 5.7

We want to prove: $\overline{OP} \equiv \overline{OQ}$

| Steps | Statement | Reason |
|---|---|---|
| 1 | $AP = \frac{AB}{2}, \quad CQ = \frac{CD}{2}$ | The perpendicular from the center bisects the chord |
| 2 | $m(\angle CQO) = m(\angle APO) = 90^0$ | $\overline{OP} \perp \overline{AB}$ and $\overline{OQ} \perp \overline{CD}$ |
| 3 | $\overline{OA} \equiv \overline{OC}$ | Radii of the same circle |
| 4 | $\overline{AB} \equiv \overline{CD}$ | Given |
| 5 | $\Delta OPA \equiv \Delta OQC$ | RHS postulate of congruency |
| 6 | $\overline{OP} \equiv \overline{OQ}$ | From statement **5** |

## Example 1

In figure 5.8, a chord of a circle of radius 5 cm is 8 cm long. Find the distance of the chord from the center.

**Solution:**

Given: $AB = 8\text{cm}$ and $OB = 5\text{cm}$

$OM^2 + MB^2 = OB^2$

$OM = \sqrt{OB^2 - MB^2}$

$= \sqrt{5^2 - 4^2}$

$= \sqrt{25 - 16}$

$= \sqrt{9}\text{ cm} = 3\text{cm}.$

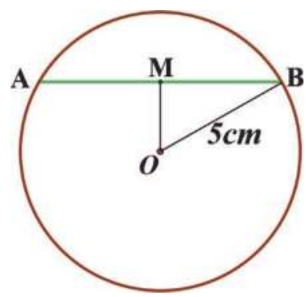


Figure 5.8

## Exercise 5.2

In figure 5.9, a chord of a circle of radius 5 cm is 6 cm long. Find the distance of the chord from the center.

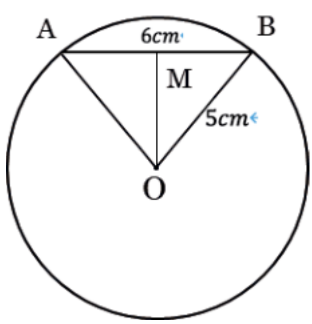


Figure 5.9

## Characteristics of Chord (2)

### Theorem 5.4

If the angles subtended by the chords of a circle are equal in measure, then the length of the chords are equal.

### Proof:

From figure 5.10, consider $\Delta AOB$ and $\Delta POQ$.

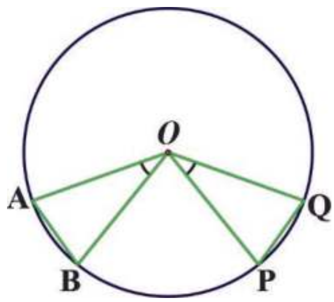



Figure 5.10

| Steps | Statement | Reason |
|---|---|---|
| 1 | $\hat{AOB} \equiv \hat{POQ}$ | Given |
| 2 | $\overline{OA} \equiv \overline{OB} \equiv \overline{OP} \equiv \overline{OQ}$ | Radii of the same circle |
| 3 | $\Delta AOB \equiv \Delta POQ$ | SAS postulate of congruence |
| 4 | $\overline{AB} \equiv \overline{PQ}$ | From step 3 |

### Theorem 5.5

Chords which are equal in length subtend equal angles at the center of the circle.

### Proof:

From figure 5.11, consider $\Delta AOB$ and $\Delta POQ$

We want to prove $\angle AOB \equiv \angle POQ$.

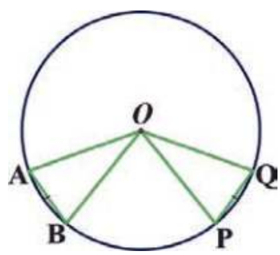



Figure 5.11

| | Statement | Reason |
|---|---|---|
| 1 | $\overline{AB} \equiv \overline{PQ}$ | Given |
| 2 | $\overline{OA} \equiv \overline{OB} \equiv \overline{OP} \equiv \overline{OQ}$ | Radii of the same circle |
| 3 | $\Delta AOB \equiv \Delta POQ$ | SSS postulate of Congruence |
| 4 | $m(\angle AOB) = m(\angle POQ)$ | From step 3 |

## Exercise 5.3

1. A chord of length 20cm is at a distance of 8cm from the center of the circle. Find the radius of the circle.
2. A chord of a circle of radius 8cm is 10cm long. Find the distance of the chord from the center of the circle.
3. AB and CD are equal chords in a circle of radius 10 cm. If each chord is 16 cm, find their distance from the center of the circle.

# 5.2 Angle Properties of Circles

## Central Angles and Inscribed Angles (1)

### Activity 5.2

1. Define the following terms: chord, diameter, radius, tangent, secant, arc.
2. Discuss a major and minor arc.

- A major arc is an arc connecting two endpoints on a circle and its measure is greater than $180°$ or $\pi$.

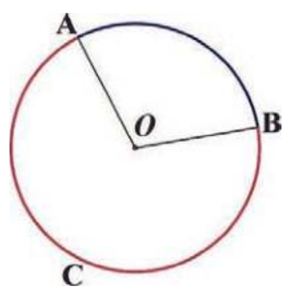



Figure 5.12

- Minor arc is an arc connecting two endpoints on a circle and its measure is less than 180° or π.
  (See figure 5.13)

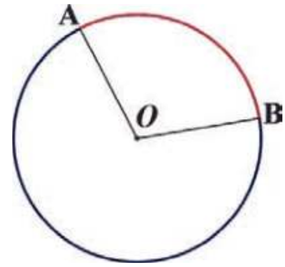



Figure 5.13

- A major arc is usually referred to with three letters and a minor arc is usually referred to with only two letters.
- A central angle is an angle formed by two radii with vertex at the center of the circle.
  (See figure 5.14)

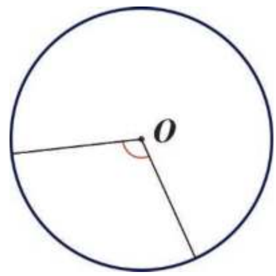



Figure 5.14

**An inscribed angle** is an angle with vertex on the circle formed by two intersecting chords.
(See figure 5.15)

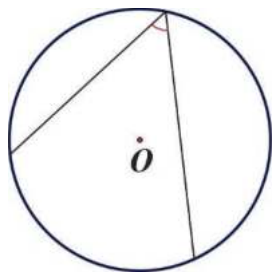



Figure 5.15

### Example 1

In figure 5.16, m($\angle AOB$) is a central angle with an intercepted minor arc $AB$ whose measure is 82°, i.e.,
$x = 82°$.

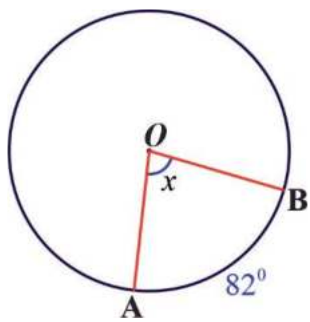



Figure 5.16

## Theorem 5.6

If an inscribed and a central angle intercept the same arc, then the measure of an inscribed angle is half of the measure of a central angle.

## Proof:

Let's prove $\theta = 2a$ for all $\theta$ and $a$, where $\theta$ is the central angle. These three cases account for all possible situations where an inscribed angle and a central angle intercept the same arc.

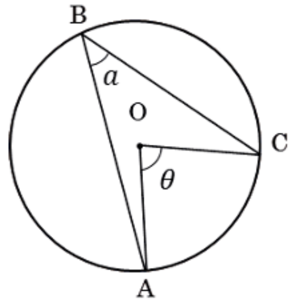



Figure 5.17

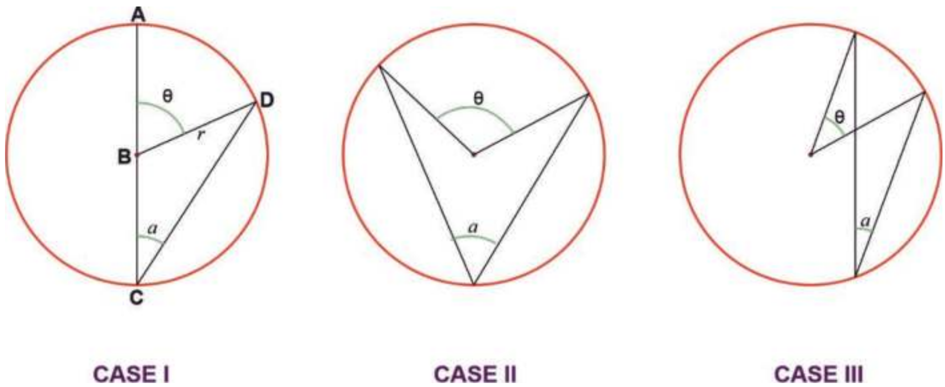



Figure 5.18

**Case I.** The diameter lies along one ray of an inscribed angle (see figure 5.19) .

**Step 1** Spot the isosceles triangle.

$\overline{BC}$ and $\overline{BD}$ are both radii, so they have the same length. So, $\Delta DBC$ is an isosceles, which also means that its base angles are congruent. Then, $m(\angle BCD) = m(\angle BDC) = a$.

**Step 2** Spot the straight angle. So, $m(\angle DBC) = 180° - \theta$

**Step 3** Write an equation and solve for $a$.

$a + a + (180° - \theta) = 180°$ which implies $2a - \theta = 0$ which again implies $2a = \theta$. Proved.

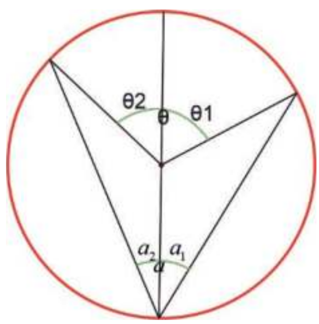



Figure 5.19

**Case II.** The diameter is between the rays of the inscribed angle $a$ (see figure 5.20).

**Step 1** Draw a diameter and using the diameter let's break '$a$' into $a_1$ and $a_2$, and $\theta$ into $\theta_1$ and $\theta_2$ as shown in figure 5.20.

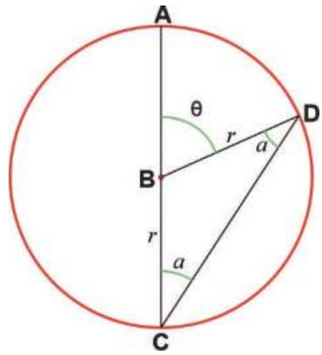


Figure 5.20

**Step 2** Use what we learned from case I to establish two equations

- $a_1 + a_1 + (180° - \theta_1) = 180°$
  $2a_1 - \theta_1 = 0$ which implies $2a_1 = \theta_1$.
- $a_2 + a_2 + (180° - \theta_2) = 180°$ which implies $\theta_2 = 2a_2$.

**Step 3** Add the above two equations

$\theta_1 + \theta_2 = 2a_1 + 2a_2$ which implies $\theta = 2(a_1 + a_2) = 2a$.

**Case III.** The diameter is outside the rays of the inscribed angle.

**Step 1** Draw a diameter and using the diameter let's create two new angles $\theta_2$ and $a_2$ as shown in figure 5.21.

**Step 2** Use what we learned from case I to establish two equations.

- $a_2 + a_2 + (180° - \theta_2) = 180°$
  $2a_2 - \theta_2 = 0$ which implies $2a_2 = \theta_2$
- $a + a_2 + a + a_2 + (180° - \theta_2 - \theta) = 180°$
  $2a + 2a_2 - \theta_2 - \theta = 0$ but $2a_2 = \theta_2$

$\therefore\ \theta = 2a$. We prove that $\theta = 2a$.

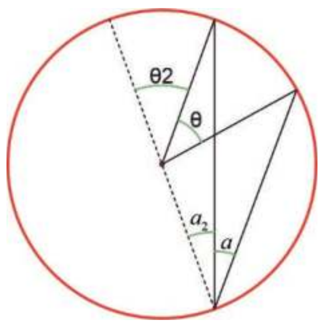


Figure 5.21

### Example 1

In the figure 5.22, if $O$ is the center of a circle with $m(\angle PRQ) = 75°$, what is the size of $\angle POQ$?

**Solution:**

$m(\angle POQ) = 2 \times m(\angle PRQ)$

$= 2 \times 75° = 150°$.

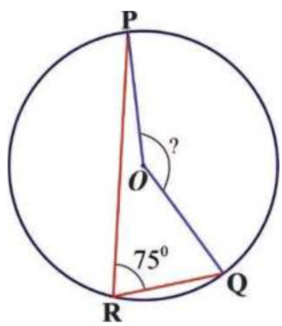


Figure 5.22

## Example 2

In figure 5.23, $\angle ABC$ is an inscribed angle with an intercepted minor arc from A to C. Find the value of $x$.

**Solution:**

$\angle ABC$ is an inscribed angle.

So, $m(\angle ABC) = \frac{1}{2}m(\text{arcAC})$

$= \frac{1}{2}(82°) = 41°$

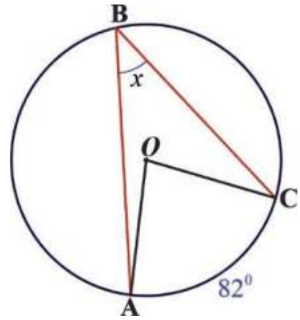



Figure 5.23

## Exercise 5.4

1. In the figure 5.24, $O$ is the center of a circle. Find the measure of $\angle ABC$.

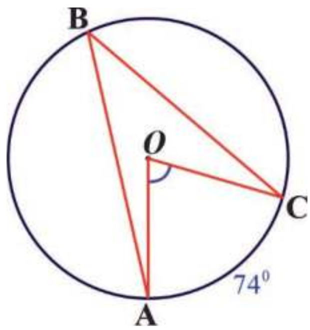



Figure 5.24

2. In figure 5.25, O is the center of a circle. Find the measure of $\angle x$.


Figure 5.25

## Central Angles and Inscribed Angles (2)

### Theorem 5.7

Inscribed angles subtended by the same arc have the same measure.

### Proof:

In Figure 5.26, $m(\angle APB) = \frac{1}{2}m(\angle AOB)$ (By theorem 5.6)

$m(\angle AQB) = \frac{1}{2}m(\angle AOB)$ (By theorem 5.6)

Therefore, $m(\angle APB) = m(\angle AQB)$.

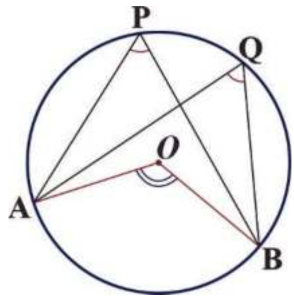



Figure5.26

### Example 1

In the figure 5.27, what is the measure of angle $CBX$?

**Solution:**

Given: $m(\angle ADB) = m(\angle ACB) = 32°$ and $\angle ACB \equiv \angle XCB$.

So, in $\Delta BXC$ we have $m(\angle BXC) = 85°$ and

$m(\angle XCB) = 32°$. By angle sum theorem,

$m(\angle CBX) + m(\angle BXC) + m(\angle BCX) = 180°$.

$m(\angle CBX) + 85° + 32° = 180°$.

$m(\angle CBX) = 180° - 117^0 = 63°$.

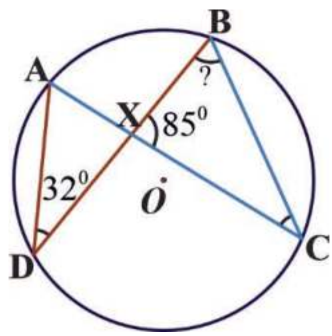



Figure 5.27

### Theorem 5.8 Angle in a semicircle (Thales' Theorem)

An angle inscribed in a semicircle is a right angle.

### Proof:

The given angle $APB$ is subtended by a semicircle as shown in figure 5.28. The corresponding central angle which is subtended by arc $AB$ is a straight angle, that is, the central angle is 180°. Hence, by theorem 5.6,

$$m(\angle APB) = \frac{1}{2}m(\angle AOB)$$

$$= \frac{1}{2} \times 180°$$

$$= 90°$$

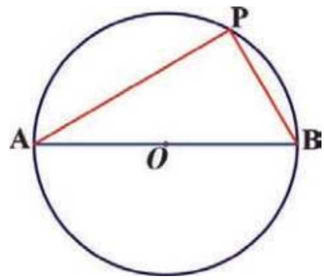



Figure 5.28

## Example 1

If $AB$ is the diameter of the circle with center $O$ as shown in the figure 5.29, then find the measure of $\angle BAC$.

**Solution:**

By Thales' Theorem $m(\angle ACB) = 90°$.

By angle sum theorem we have,

$m(\angle BAC) + m(\angle ACB) + m(\angle ABC) = 180°$,

$m(\angle BAC) + 90° + 55° = 180°$.

Hence $m(\angle BAC) = 35°$.

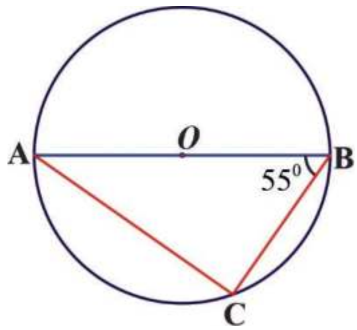



Figure 5.29

## Exercise 5.5

In figure 5.30, O is the center of a circle. Find the measure of $\angle x$ and $\angle y$.

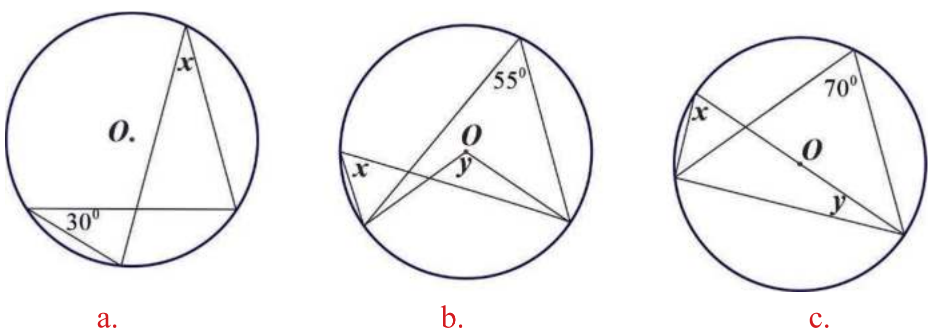



Figure 5.30

## Cyclic Quadrilateral

### Definition 5.1

A quadrilateral is said to be a cyclic quadrilateral if there is a circle passing through all its four vertices.

### Theorem 5.9

In a cyclic quadrilateral WXYZ in figure 5.31, the sum of either pair of opposite angles is $180^\circ$.

### Proof:

Given: A cyclic quadrilateral WXYZ is inscribed in a circle with center $O$ as shown in figure 5.31.

Construction: Join the vertices $W$ and $Y$ with center $O$.

We want to show: m(∠WXY) + m(∠ WZY) = 180°.

Consider arc WXY and arc WZY

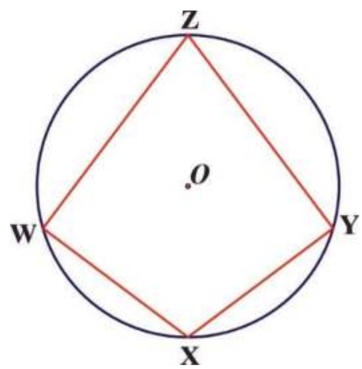


Figure 5.31

1. ∠WOY ≡ 2∠WZY (The angle subtended by same arc is half of the angle subtended at the center)
2. Reflex angle WOY ≡ 2arcWXY (the angle subtended by same arc is half of the angle subtended at the center)
3. m(∠WOY) + Reflex m(∠WOY) ≡ 360° (Using steps 1 and 2)
4. 2m(∠WZY) + 2m(∠WXY) = 360° (Using steps 1 and 2)
5. 2(m(∠WZY) + m(∠WXY)) = 360°(Why?)
6. m(∠WZY) + m(∠WXY) = 180° (Why?)

### Example 1

If the measures of all four angles of a cyclic quadrilateral are given as $(4y + 2)$, $(y + 20)$, $(5y - 2)$, and $7y$ respectively, find the value of $y$.

**Solution:**

The sum of all four angles of a cyclic quadrilateral is 360°. So, to find the value of $y$, we need to equate the sum of the given four angles to 360°.

$$(4y + 2) + (y + 20) + (5y - 2) + 7y = 360°$$
$$17y + 20 = 360°$$
$$17y = 340°.$$
$$\text{Therefore, } y = 20.$$

## Exercise 5.6

1. In figure 5.32, $ABCD$ is a cyclic quadrilateral drawn inside a circle with center $O$ and $m(\angle ABC) = 108°$. What is the measure of $m(\angle ADC)$?

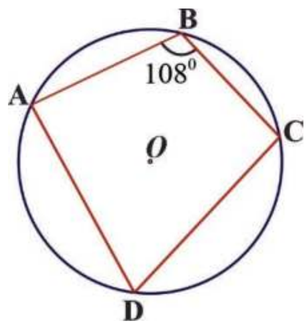


Figure 5.32

2. In the figure 5.33, $O$ is the center of a circle. Find the measure of $\angle x$ and $\angle y$.

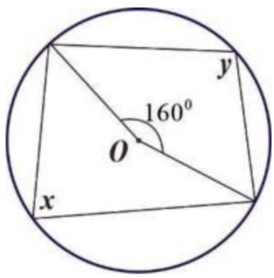


Figure 5.33

3. Find the value of angle $x$ and angle $y$ as shown in figure 5.34.

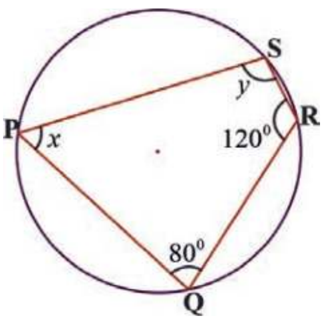


Figure 5.34

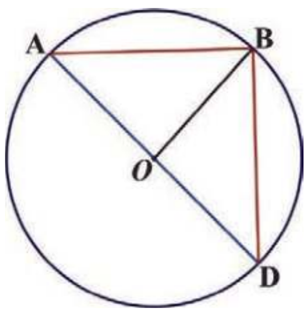


Figure 5.35

4. Given circle with center O, as shown in the figure 5.35,
   a. Name one minor arc in the circle.
   b. Name one major arc in the circle.
   c. Name the angle subtended by arc BD.
   d. Name the inscribed angle subtended by arc BD.
   e. Name an angle in a semicircle.
   f. Two angles subtended by chord $AB$.
5. Given: Circle with center $O$ as shown in the Figure 5.36 with $m(\angle DOB) = 86°$.
   Calculate:
   a. $m(\angle DAB)$
   b. $m(\angle AOB)$
   c. $m(\angle ABD)$
   d. $m(\angle ODB)$
   e. $m(\angle AEB)$

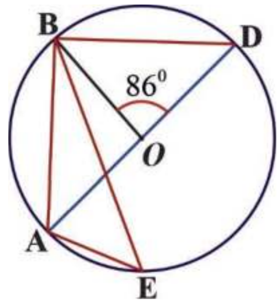



Figure 5.36

6. Let V, W, X and Y are points on the circumference of a circle with center $O$ as shown in figure 5.37. Chords VX and WY intersect at a point Z, $(\angle XVW) = 72°$ and $m(\angle VXY) = 28^0$. What is the measure of $(\angle VZW)$?

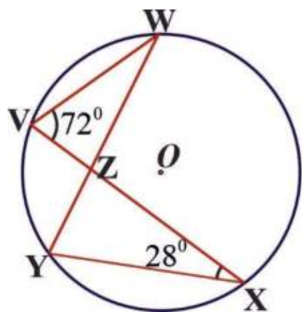
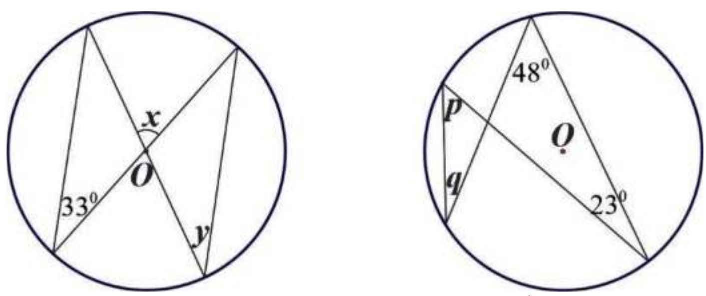



Figure 5.37

7. In each of the following figures, $O$ is the center of the circle. Calculate the measure of the angles marked $x, y, p$ and $q$.




a. b.

Figure 5.38

# 5.3 Arc Lengths, Perimeters and Areas of Segments and Sectors

## Length of Arc and Chord

**Activity 5.3**

1. Discuss circumference and central angle.
2. Define a sector.

Arc length is an important aspect to understand portions of curved lengths. As you will learn in this lesson, combining our knowledge of circumference and central angle measures, we will find arc length.

**Definition 5.2**

**Arc length** is the length of an arc which is a portion of the circumference of a circle.

For a circle of radius r subtended by an angle $\theta$, the length $s$ of the corresponding arc is:

$$s = 2\pi r \times \frac{\theta}{360^0} \quad \text{or} \quad \pi d \times \frac{\theta}{360^0}$$

where $d$ is the diameter of a circle (see figure 5.39).

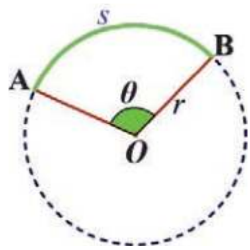



Figure 5.39

**Example 1**

Find the arc length that a central angle of 150° subtends in a circle of radius 6 cm as shown in figure 5.40.

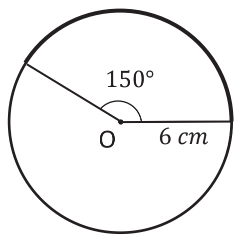



Figure 5.40

**Solution:**

The length $s$ of the corresponding arc is

$$s = 2\pi r \times \frac{\theta}{360^0}$$

Given: $\theta = 150°, r = 6\text{cm}.$

arc length$= s = 2\pi r \times \frac{\theta}{360^0} = 2\pi \times 6 \times \frac{150°}{360°} = 5\pi$ cm

## Length of a chord

Let the midpoint of $AB$ be $M$, and the length of the chord $AB$ and $AM$ be $l$ and $m$ respectively. Using trigonometric ratios on the right-angled triangle OAM,

$$m = r\sin\left(\frac{\theta}{2}\right)$$

Then, $l = 2m = 2r\sin\left(\frac{\theta}{2}\right)$

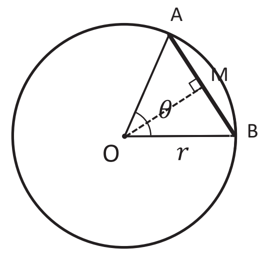



Figure 5.41

In general, the length of a chord is defined as $l = 2r\sin\left(\frac{\theta}{2}\right)$, where $r$ is the radius of the circle and $\theta$ is the angle subtended at the center by the chord.

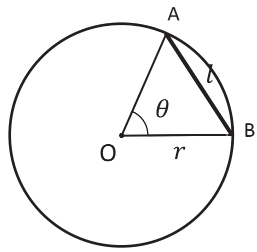



Figure 5.42

### Example 2

Find the chord length $AB$ if $\theta = 120°$ when a central angle subtended in a circle of radius 8cm as shown in figure 5.43.

**Solution:**

Chord length $= 2r\sin\left(\frac{\theta}{2}\right) = 2 \times 8\text{cm} \times \sin\left(\frac{120°}{2}\right)$

$= 16\text{cm} \times \sin(60°)$

$= 16\text{cm} \times \frac{\sqrt{3}}{2} = 8\sqrt{3}\text{cm}$

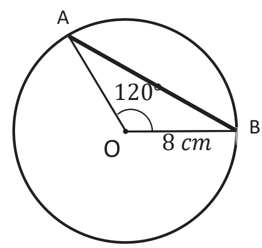



Figure 5.43

## Perimeter of a segment

A **segment** is part of a circle bounded in between a chord and an arc of a circle.

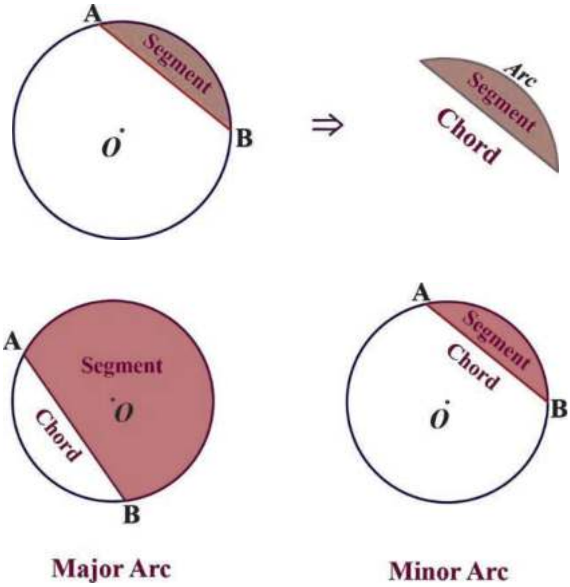



Figure 5.44

Inside a circle, there can be a minor segment and a major segment as shown in figure 5.44. A segment has an area and perimeter that measures all the way around the edge of the segment.

- When a segment in a circle is bounded by a chord and an arc, the perimeter of a segment is given by:

$$\text{Perimeter of segment} = \text{Length of chord } + \text{Length of arc}$$

### Example 1

Find the perimeter of the segment as shown in figure 5.45. (Use π.)

**Solution:**

Perimeter of segment $= 2\pi \times 10 \times \frac{160°}{360°} + 2 \times 10 \times \sin\left(\frac{160°}{2}\right)$

$= 20\pi \times \frac{4}{9} + 20 \times \sin(80°)$

$= \frac{80}{9}\pi + 20 \times 0.985$

$= \left(\frac{80}{9}\pi + 19.7\right)$ m

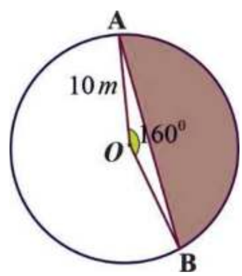



Figure 5.45

## Exercise 5.7

1. Name the parts of circles in figure 5.46.

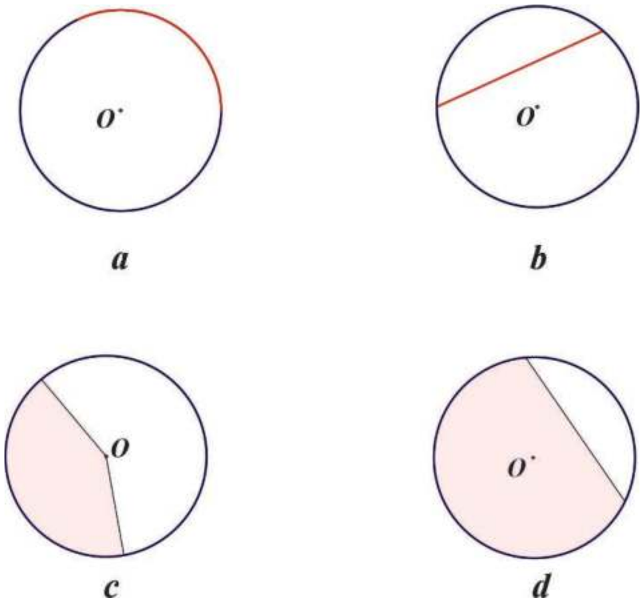
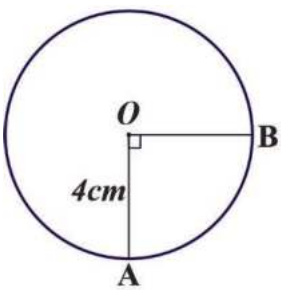
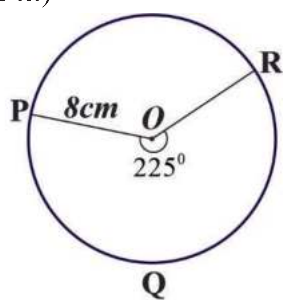



Figure 5.46

2. Find the length of arc AB and arc PQR. (Use π.)

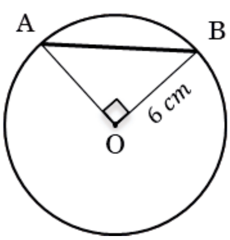


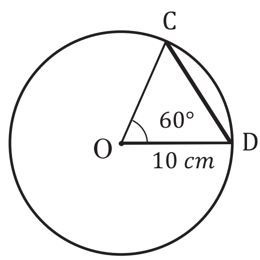

a.



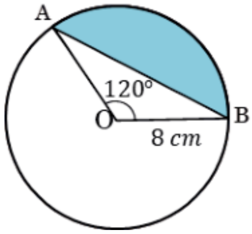

b.

3. Find the length of chords AB and CD.



4. Find the perimeter of the shaded segment with radius 8cm. (Use $\pi$.)




## Area and Perimeter of Sector

The **sector** is basically a portion of a circle enclosed by two radii and an arc. It divides the circle into two regions, namely major and minor Sector

(see figure 5.47).

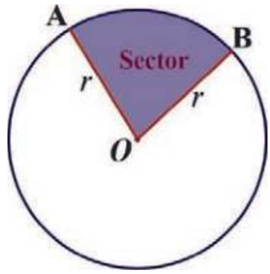



Figure 5.47

## Area of a Sector

Consider a circle of radius $r$, centre $O$, and

$m(\angle POQ) = x = \theta$

(in degrees) as shown in figure 5.48. The area of a sector is given by:

$$A_{sector} = \pi r^2 \left(\frac{\theta}{360°}\right)$$

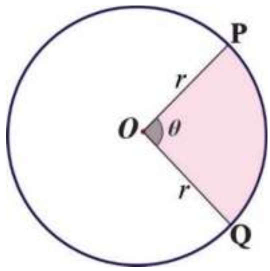



Figure 5.48

## Perimeter of a Sector

Perimeter of sector $= 2$ radius $+$ arc length.

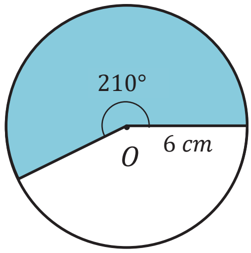
Therefore, the perimeter of sector $= 2r + 2\pi r \times \frac{\theta}{360°}$

### Example 1

If the angle of the sector with radius 6 cm is $210°$, then find the area and perimeter of the sector. (Use $\pi$.)

**Solution:**

Given: $r = 6$ cm, $\theta = 210°$




$$A_{sector} = \pi r^2 \left(\frac{\theta}{360°}\right)$$
$$= \pi \times (6)^2 \times \left(\frac{210°}{360°}\right)$$
$$= \pi \times 36 \times \frac{7}{12}$$
$$= 21\pi \text{ cm}^2$$

$$P_{sec} = 2r + 2\pi r \times \frac{\theta}{360°}$$
$$= 2 \cdot 6 + 2\pi \cdot 6 \times \frac{210°}{360°}$$
$$= 12 + 7\pi$$

Thus, the perimeter of the sector is $(12 + 7\pi)$cm.

## Area of a Segment

An arc and two **radii** of a circle form a **sector**. These two radii and the chord of the segment together form a triangle. Thus, the area of a segment of a circle is obtained by subtracting the **area of the triangle** from the area of the sector. i.e., Area of a segment of circle = area of the sector − area of the triangle.

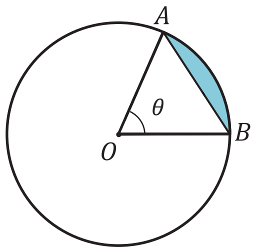



Figure 5.49

$$A = \pi r^2 \times \frac{\theta}{360°} - \frac{1}{2} r^2 \sin\theta$$

### Example 2

Find the area of the shaded region as shown in the figure 5.50.

**Solution:**

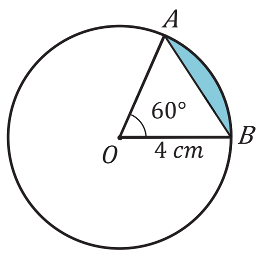



Figure 5.50

Area of segment: $A = \pi r^2 \times \frac{\theta}{360°} - \frac{1}{2} r^2 \sin\theta$

$$= \pi \cdot 4^2 \times \frac{60°}{360°} - \frac{1}{2} \times 4^2 \times \sin 60°$$
$$= 16 \cdot \frac{1}{6}\pi - \frac{1}{2} \cdot 16 \cdot \frac{\sqrt{3}}{2}$$
$$= \frac{8}{3}\pi - 4\sqrt{3}$$

Hence, the area is $(\frac{8}{3}\pi - 4\sqrt{3})\ \text{m}^2$.

### Example 3

If the area of a sector is 100 $\text{m}^2$ and the area of the enclosed triangle is 78 $m^2$, what is the area of the segment?

**Solution:**

Area of the segment = area of the sector − area of the triangle

$$= 100 - 78$$

$$= 22$$

Therefore, the area of the segment is 22 $\text{m}^2$.

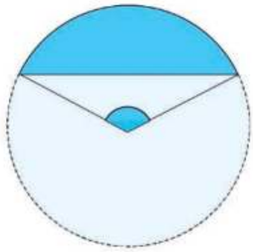


Figure 5.51

## Exercise 5.8

1. Find the area and perimeter of the shaded sectors (use $\pi$).

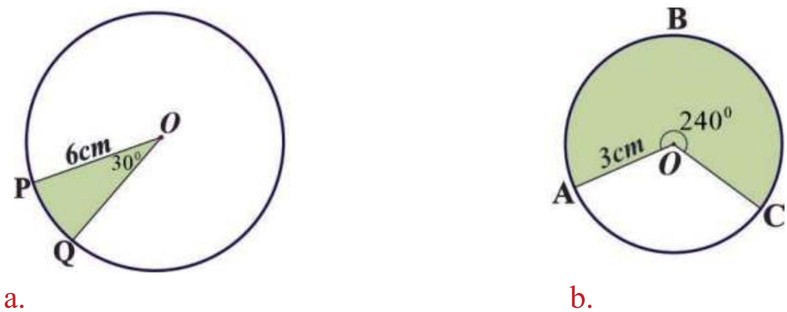



Figure 5.52

2. Find the area of the segment (use $\pi$).


Figure 5.53

# 5.4 Theorems on Angles and Arcs Determined by Lines Intersecting inside, on and outside a Circle.

## Angles Formed by Chords

### Theorem 5.10

If two chords intersect inside a circle, then the measure of an angle formed between the chords is one half the sum of the measure of the arcs intercepted by the angle and its vertical angle.

In the circle, as shown in figure 5.54, the two chords PR and QS intersect inside the circle at point T.

$$m(\angle PTQ) = \frac{1}{2}[m(arcPQ) + m(arcRS)]$$

$$\text{and } m(\angle QTR) = \frac{1}{2}[m(arcQR) + m(arc(PS)]$$

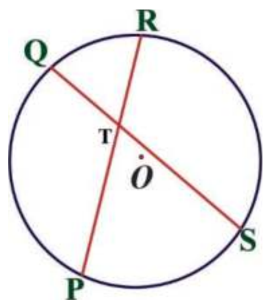



Figure 5.54

### Example 1

In the circle shown in figure 5.55 if $m(arcPQ) = 68°$ and $m(arcRS) = 128°$, then find measure of $\angle RTS$, where $\bar{PR}$ and $\bar{QS}$ intersects at $T$.

**Solution:**

$$m(\angle RTS) = \frac{1}{2}[m(arcPQ) + m(arcRS)]$$

$$= \frac{1}{2}(68° + 128°)$$

$$= \frac{1}{2} \times 196° = 98°$$

Therefore, $m(\angle RTS) = 98°$

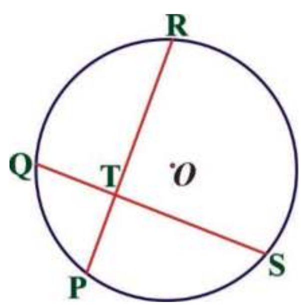



Figure 5.55

## Exercise 5.9

1. In figure 5.56, if $m(arcAC) = 28°$ and $m(arcBD) = 132°$, find $x$.

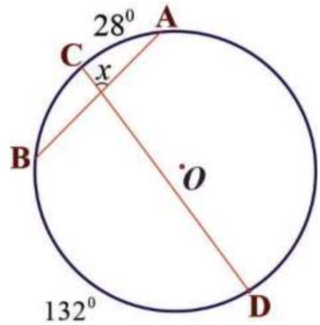



Figure 5.56

2. Given a circle with center $O$, as shown in figure 5.57. If $m(\text{arc } HT) = m(\text{arc } JH)$ and $m(\angle PTH) = 58^0$ , calculate the measure of the remaining angles $\angle PRH, \angle THS, \angle TSH, \angle HTS$ and minor arc $HT$, minor arc $TS$.

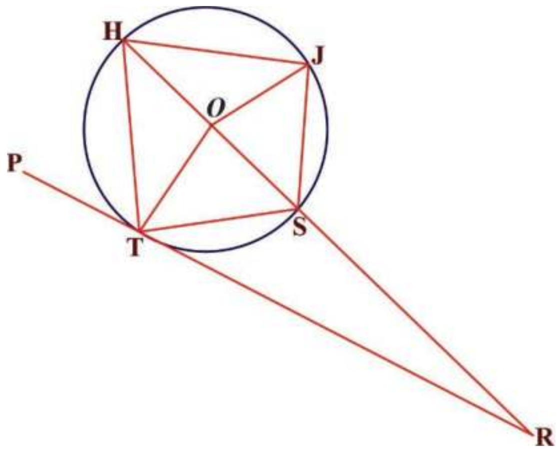



Figure 5.57

3. Prove Theorem: The point of intersection separates the chord into 2 segments. The product of the lengths of the segments for one chord is the same as the product for the other chord, i.e., $\overline{AP} \cdot \overline{PB} = \overline{CP} \cdot \overline{PD}$ (see figure 5.58).

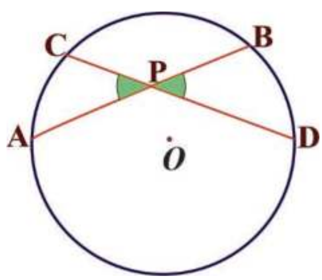



Figure 5.58

4. Let's prove theorem 5.10 following the steps below.

**Step 1.** Draw four radii $OP, OQ, OR$ and $OS$ and chord $QR$.

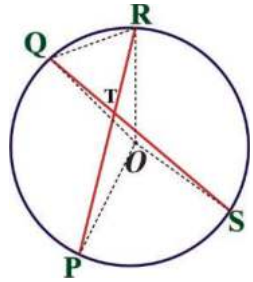



Figure 5.59

**Step 2.** Show $\text{m}(\angle QRT) = \frac{1}{2}\text{m}(\angle QOP)$. Similarly, show $\text{m}(\angle RQT) = \frac{1}{2}\text{m}(\angle ROS)$.

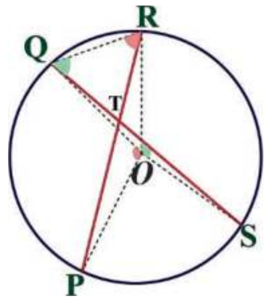



Figure 5.60

**Step 3.** Show $\text{m}(\angle QTP) = \text{m}(\angle QRT) + \text{m}(\angle RQT)$.

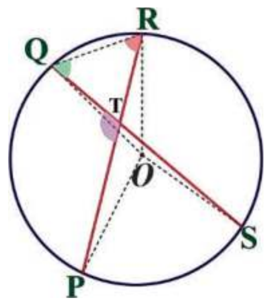



Figure 5.61

**Step 4.** From step 2 and 3,

show $\text{m}(\angle QTP) = \text{m}(\angle QRT) + \text{m}(\angle RQT)$

$$= \frac{1}{2}\text{m}(\angle QOP) + \frac{1}{2}\text{m}(\angle ROS)$$

$$= \frac{1}{2}[\text{m}(\angle QOP) + \text{m}(\angle ROS)]$$

$$= \frac{1}{2}\text{m}(arcQP + arcRS)$$

## Angles Formed by Secants and Tangents

### Theorem 5.11

The measure of the angle formed by two secants, two tangents, or a secant and a tangent that intersect at a point outside a circle is equal to one-half the positive difference of the measures of the intercepted arcs.

In figure 5.62, we illustrate this result for the angle formed by the intersection of two secants, $\overline{AC}$ and $\overline{AE}$ .

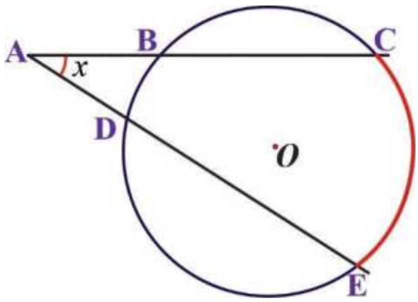



Figure 5.62

The minor arc intercepted by the two secants is $BD$ and an arc $CE$. Hence, by the theorem of angles between intersecting secants,

$$x = \frac{1}{2}[\mathrm{m}(\mathrm{arc}CE) - \mathrm{m}(\mathrm{arc}BD)]$$

In the same way, we illustrate the result for the intersection of two tangents $\overline{AB}$ and $\overline{AC}$ as shown in figure 5.63.

Hence, $x = \frac{1}{2}[\mathrm{m}(\mathrm{arc}BDC) - \mathrm{m}(\mathrm{arc}BC)]$

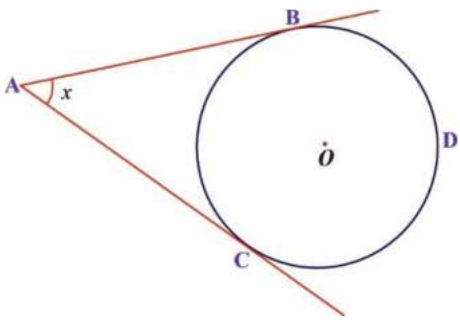



Figure 5.63

### Example 1

In figure 5.64, if m(arc$BD$) = 43° and m(arc$CE$) = 107°, then find m(∠CAE).

**Solution:**

$$m(\angle CAE) = \frac{1}{2}(m(arcCE) - m(arcBD))$$
$$= \frac{1}{2}(107° - 43°)$$
$$= \frac{1}{2}(64°) = 32^0$$

Therefore, $m(\angle CAE) = 32^0$

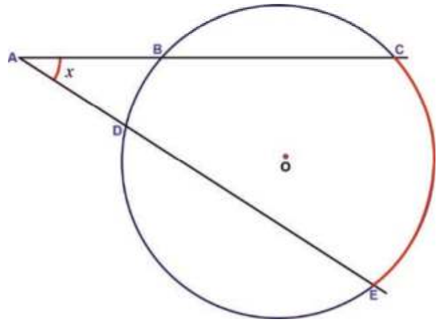


Figure 5.64

## Exercise 5.10

1. In figure 5.65, $\overline{RS}$ and $\overline{RT}$ are tangent lines to the circle with center $O$. If ∠$SRT$) = 40°, what is the m(∠$TUS$)?

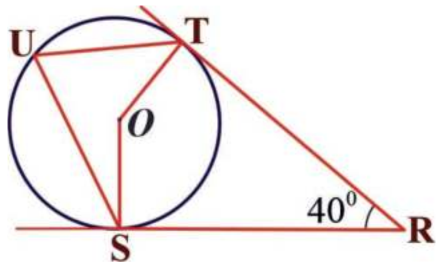


Figure 5.65

2. Theorem: The measure of an angle inscribed in a circle is half of the measure of the arc subtending. Prove it (see figure 5.66).

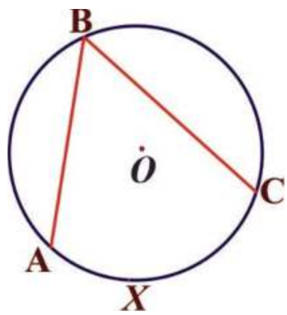


Figure 5.66

3. Theorem: An angle inscribed in a semi-circle is a right angle (see figure 5.67). The converse of this corollary is that a circular arc in which a right angle is inscribed must be a semi-circle, i.e.,

$m(\angle ABC) = \frac{1}{2}m(arcAXC) = 90°.$

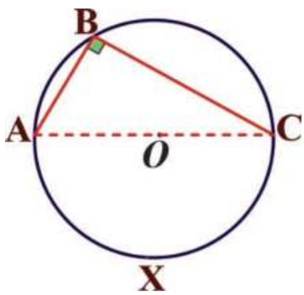


Figure 5.67

4. Discuss a tangent line to a circle and point of tangency.
5. Prove that if from one external point, two tangents are drawn to a circle then they have equal tangent segments.
6. Let's prove theorem 5.11 following the steps below.

**Step 1.** Draw four radii $OB, OC, OE$ and $OD$.

Make $m(\angle COE) = a$ and $m(\angle BOD) = b$

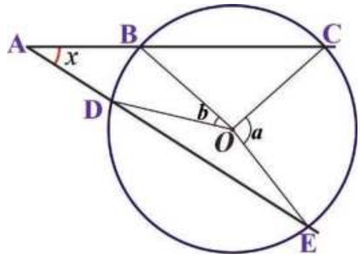



Figure 5.68

**Step 2.** Draw two chords BE and DC.

Then $m(\angle CBE) = \frac{1}{2}m(\angle COE) = \frac{1}{2}a.$

Similarly, $m(\angle BED) = \frac{1}{2}m(\angle BOD) = \frac{1}{2}b.$

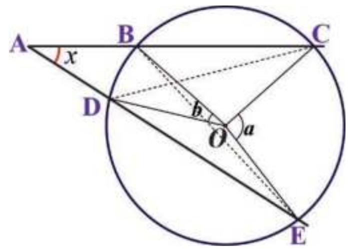



Figure 5.69

**Step 3.** Take $\Delta ABE$. $m(\angle BAE) + m(\angle BEA) = m(\angle CBE)$

**Step 4.** From step 3, show $x + \frac{1}{2}b = \frac{1}{2}a.$

Therefore,

$x = \frac{1}{2}a - \frac{1}{2}b = \frac{1}{2}(a - b).$

$\therefore m(\angle CAE) = \frac{1}{2}[m(\angle COE) - m(\angle BOD)]$

$= \frac{1}{2}[m(\text{arc}CE) - m(\text{arc}BD)].$

# Summary

1. A circle is the locus of points (set of points) in a plane each of which is equidistant from a fixed point in the plane.
2. A tangent line to a circle meets the circle in one point and is perpendicular to the radius (and diameter) to that point.
3. The line segment joining the center of a circle to the midpoint of a chord is perpendicular to the chord.
4. A central angle is an angle formed by two radii with vertex at the center of the circle.
5. An inscribed angle is an angle with its vertex on the circle formed by two intersecting chords.
6. A figure has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half. A figure that has one line of symmetry is called a symmetrical figure.
7. A diameter perpendicular to a chord bisects the chord.
8. The perpendicular bisector of a chord passes through the center of the circle.
9. In the same circle, equal chords are equidistant from the center.
10. Line segments that are tangents to a circle from an outside point are equal.
11. If an inscribed and a central angle intercept the same arc, then the measure of the inscribed angle is half of the measure of the central angle.
12. The angle inscribed in a semicircle is a right angle.
13. The length $l$ of an arc that subtends an angle $\theta$ at the center of the circle with radius $r$ is:

$$l = 2\pi r \times \frac{\theta}{360^0}$$

14. The length $m$ of a chord that subtended an angle $\theta$ at the center of the circle with radius $r$ is: $m = 2r\sin(\frac{\theta}{2})$

**15.** The area $A$ of a sector with central angle $\theta$ and radius $r$ is given by:

$$A = \pi r^2 \times \frac{\theta}{360°}$$

**16.** The area $A$ of a segment associated with a central angle $\theta$ and radius $r$ is given by:

$$A = \pi r^2 \times \frac{\theta}{360°} - \frac{1}{2} r^2 \sin\theta$$

## Review Exercise

1. Two chords, $\overline{PQ}$ and $\overline{RS}$, of a circle intersect at right angles at a point inside the circle and if $m(\angle QPR) = 25°$, find $m(\angle PQS)$.

2. In figure 5.70, AB and CD are chords of circle $O$. Line segment of OP is perpendicular to AB and line segment of OQ is perpendicular to CD. If the lengths of line segment of OP and OQ are equal, prove that the lengths of chords of AB and CD are equal.

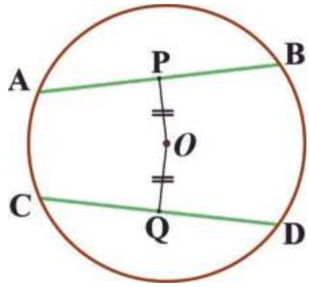



Figure 5.70

3. Find $x$ and $y$ from figure 5.71.

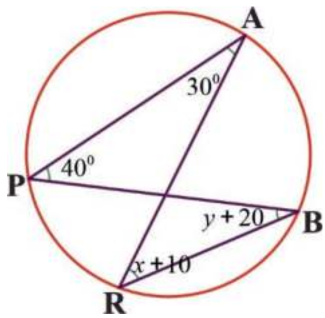



Figure 5.71

4. If $m(arcPXQ) = 168°$ and the chord and the tangent intersect at the point of tangency as shown in figure 5.72, then find $y$.

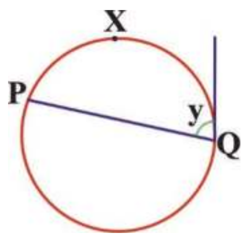



Figure 5.72

5. If two chords intersect inside the circle as shown in figure 5.73, then find $x$.

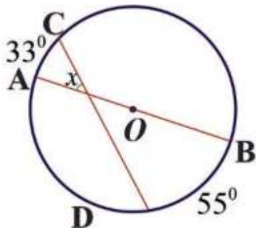



Figure 5.73

**6.** If $L$, $M$ and $N$are points on the circumference of a circle with center $O$ and $m(\angle MON) = 98°$, as shown in figure 5.74, then find $m(\angle MLN)$

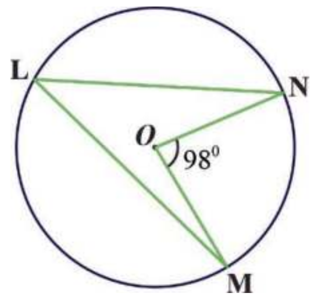



Figure 5.74

**7.** In the figure 5.75, if m (arc $AC$) = m (arc $CF$),

m (arc $AB$) = $46°$, $m(\angle AGF) = 58°$,

$m(\angle BCD) = 10°$ and m $(\angle GCF) = 35°$.

Calculate:
**a.** $m(\angle ABC)$
**b.** $m(arcDG)$
**c.** $m(\angle DHG)$

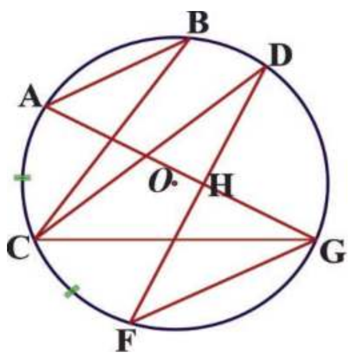



Figure 5.75

**8.** Theorem: Prove that sum of the opposite angles of a cyclic quadrilateral is $180°$.

**9.** Find the area of the segment shown in figure 5.76 if the central angle is 0.6 rad and the radius is 10 m.

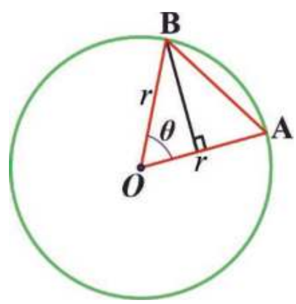



Figure 5.76

**10.** In figure 5.77, $\overline{AP}$ is tangent to the circle. Prove that $\angle ACP \equiv \angle BAP$

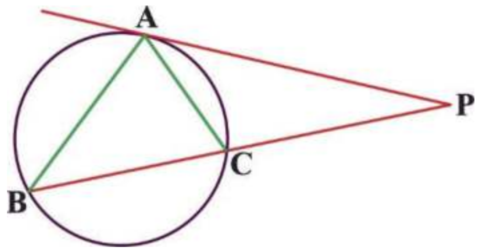



Figure 5.77

**11.** In figure 5.78, what is the measure of arc ADC If $m(\angle ABC) = 120°$?

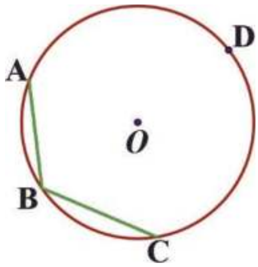



Figure 5.78

**12.** In figure 5.79, $ABCD$ is a cyclic parallelogram. Show that it is a rectangle.

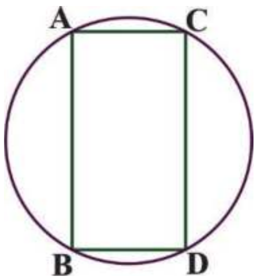



Figure 5.79

**13.** In each of the following figures, $O$ is the center and $\overline{AB}$ is the diameter of the circle. Calculate the value of $x$ in each case.

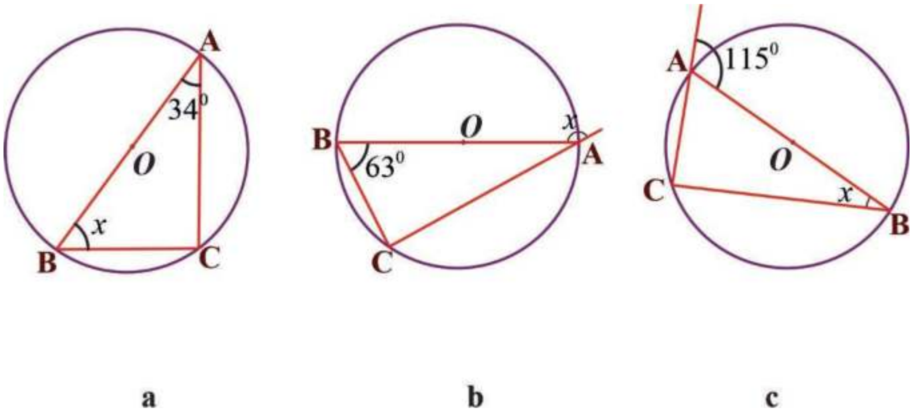



Figure 5.80

**14.** In figure 5.81, $O$ is the Centre of a circle, $m(\angle AOB) = 80°$ and $m(\angle PQB) = 70°$. Find $m(\angle PBO)$

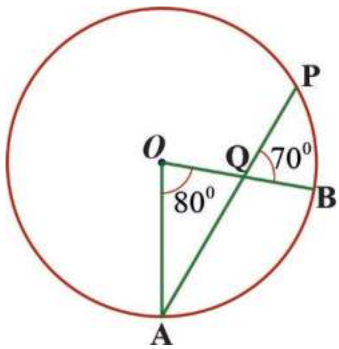



Figure 5.81

# UNIT 6

# SOLID FIGURES

## Unit Outcomes

By the end of this unit, you will be able to

- **Find surface area and volume of pyramids and cones.**
- **Calculate volume of frustum of pyramids and cones.**

## Unit Contents

**KEY Words**

- ✓ Lateral surface
- ✓ Slant height
- ✓ Prism
- ✓ Sphere
- ✓ Pyramid
- ✓ Total surface Area
- ✓ Frustum
- ✓ Regular pyramid
- ✓ Volume
- ✓ Cone
- ✓ Cross-section
- ✓ Cylinder

## INTRODUCTION

Recall that **solid figures** are three-dimensional objects, meaning they have length, width and height. Because they have three dimensions, they have depth and take up space in our universe. Based on your lower grade lessons, you are familiar with solid figures like cylinders, prisms, pyramids and cones. You have also seen the formulas to find the surface areas and volumes of these solid figures. In this unit, you will learn more about these solid figures. You will study surface areas and volumes of spheres, frustum of pyramids, frustum of cones and composite solids.

## 6.1 Revision of Prisms and Cylinders

From previous grades you learned about prisms and cylinders. You also studied the formulas for finding their surface areas and volumes.

### Activity 6.1

1. Complete the blank space for the triangular prism shown in Figure 6.1.
   a. Indicate the bases of the prism.
   b. The region ABED is called ____________.
   c. $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are called _____________.
   d. _________ is the altitude of the prism.

e. If $\overline{CF}$ were perpendicular to the plane of the triangle DEF then the prism would be called ___________.

f. The perimeter of the region DEF is the sum of _________________.

2. Sketch a right rectangular prism.

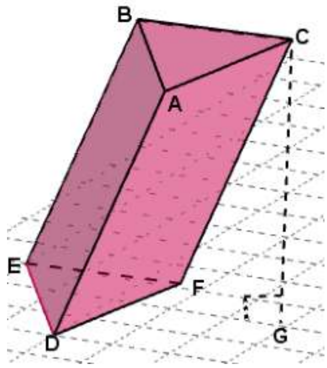



Figure 6.1

If LSA denotes the lateral surface area, BA denotes the base area and TSA denotes the total surface area of a right prism, then

LSA = $ph$, where $p$ represents the perimeter of the base region and $h$ represents the altitude of the prism,

TSA = LSA + 2BA.

The volume $V$ of any prism equals the product of its base area BA and altitude $h$. That is, V = BA·h

## Example 1

Find the total surface area and the volume of the right triangular prism shown in figure 6.2.

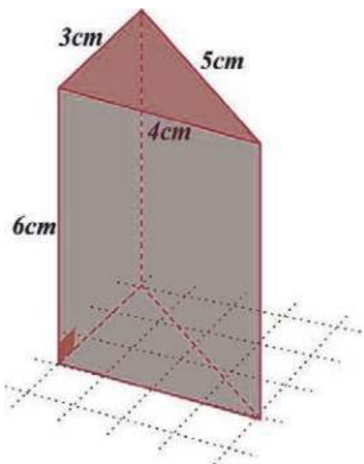



Figure 6.2

### Solution:

LSA = $ph = (4 + 3 + 5) \times 6 = 72 \text{ cm}^2$.

Since the triangle has sides of lengths 3 cm, 4 cm and 5 cm it is a right-angled triangle with legs 3 cm and 4 cm and hypotenuse 5 cm.

Observe also that $3^2 + 4^2 = 5^2$.

Therefore,

$BA = \frac{1}{2}ab = \frac{1}{2} \times 3 \times 4 = 6 \text{ cm}^2$,

$TSA = LSA + 2BA = 84 \text{ cm}^2$.

Volume $V = BA \cdot h = 6 \times 6 = 36 \text{ cm}^3$.

## Example 2

Find the total surface area and volume of the rectangular prism shown in figure 6.3.

**Solution:**

$LSA = ph = (2 \times 2 + 2 \times 3) \times 4 = 40 \text{ cm}^2$,

$BA = lw = 3 \times 2 = 6 \text{ cm}^2$,

$TSA = LSA + 2BA = 52 \text{ cm}^2$.

Volume $V = BA \cdot h = 6 \times 4 = 24 \text{ cm}^3$.

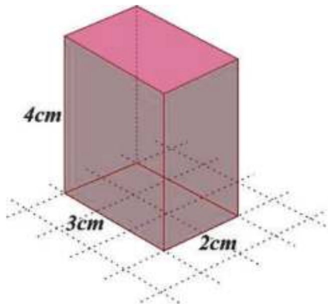


Figure 6.3

## Example 3

Find the total surface area and volume of the prism shown in figure 6.4.

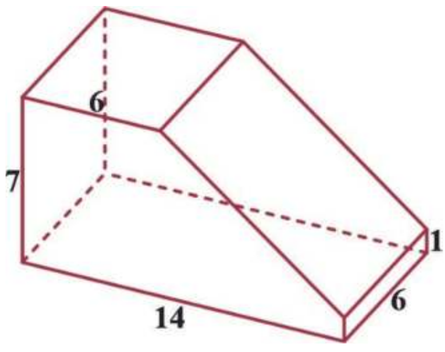


Figure 6.4

**Solution:**

The solid in Figure 6.4 is a prism with base as shown by the shaded region in figure 6.5. Base area is the sum of areas of the rectangle and the trapezium.

$BA = (7 \times 6) + \frac{1}{2}(7 + 1)(8) = 74$, Since $\overline{BD}$ is the hypotenuse of the right-angle triangle $\Delta BCD$, the length of

$\overline{BD} = \sqrt{6^2+8^2} = 10$,

LSA = $ph = (7 + 14 + 1 + 10 + 6) \times 6$

$= 228$,

TSA = LSA + 2BA = 376.

V = BA $\cdot\, h = 74 \times 6 = 444$.

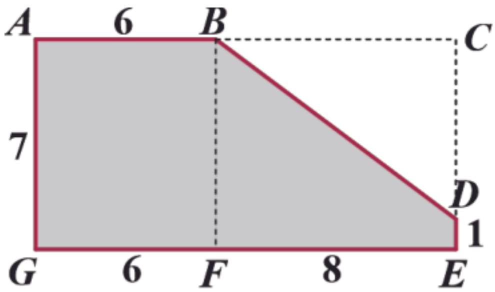
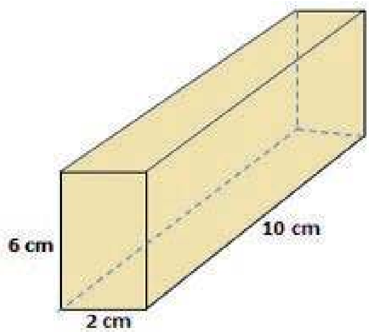



Figure 6.5

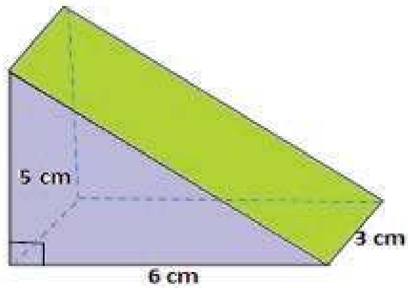
## Exercise 6.1

1. Find the total surface area and volume of the following solid figures.

a.




b.




c.

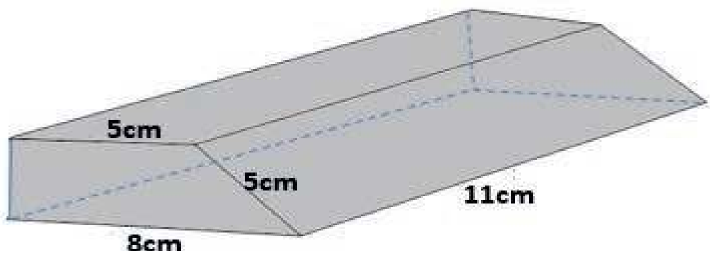



Figure 6.6

2. The base of a right prism is an equilateral triangle with a side of 4 cm and its height is 10 cm. Find its total surface area and volume.
3. Find the perimeter of the base of a right prism for which the area of the lateral surface is 120 $cm^2$ and for which the altitude is 5 cm.

4. Find the total surface area and volume of a cube of edge $s$ in length.

## Activity 6.2

Complete the blank space for the circular cylinder shown in Figure 6.7.

a. The region C is called ______________________.

b. The regions $A_1$ and $A_2$ are called ____________________.

c. Are the regions $A_1$ and $A_2$ congruent and parallel?

d. The altitude of the cylinder is ________.

e. If $\overline{LM}$ were perpendicular to the plane of the region $A_1$ then the cylinder would be called ___________.

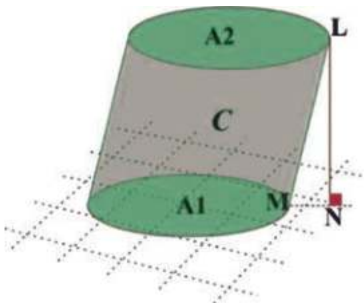



Figure 6.7

The lateral surface area of a right circular cylinder is the product of the circumference of the base and altitude of the cylinder. That is,

LSA = $2\pi rh$ where $r$ is the radius of the base and $h$ is the altitude of the cylinder.

The total surface area is the sum of the areas of the bases and the lateral surface area. That is,

$$\text{TSA} = \text{LSA} + 2\text{BA}$$

$$\text{TSA} = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$$

The volume V of a circular cylinder is the product of its base area BA and altitude $h$. That is,

$V = BA \cdot h$

$V = \pi r^2 h$, where $r$ is the radius of the base.

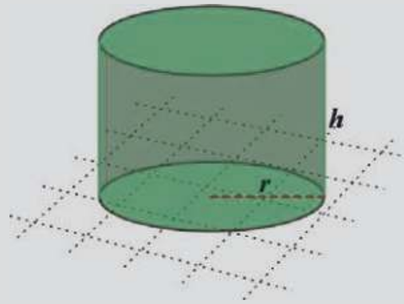



Figure 6.8

## Example 4

Find the total surface area and volume of a right circular cylinder whose radius is 3 cm and whose altitude is 5 cm.

**Solution:**

$$TSA = LSA + 2BA = 2\pi r(h + r) = (2\pi \times 3)(5 + 3) = 48\pi \text{ cm}^2.$$

$$V = \pi r^2 h = 45\pi \text{ cm}^3.$$

## Example 5

Find the total surface area and volume of the solid figure in figure 6.9.

**Solution:**

The bases are annulus regions with inner radius 1cm and outer radius 4cm. Let the inner radius be $r$ and the outer radius be R. Then,

BA = Annulus area $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

$= \pi(4^2 - 1^2) = 15\pi \text{ cm}^2.$

Outer LSA$= 2\pi Rh = 2\pi \times 4 \times 6 = 48\pi \text{ cm}^2.$

Inner LSA $= 2\pi rh = 2\pi \times 1 \times 6 = 12\pi \text{ cm}^2.$

TSA = LSA + 2BA $= 60\pi + 30\pi = 90\pi \text{ cm}^2.$

Volume = BA $\cdot\, h = 15\pi \times 6 = 90\pi \text{ cm}^3.$

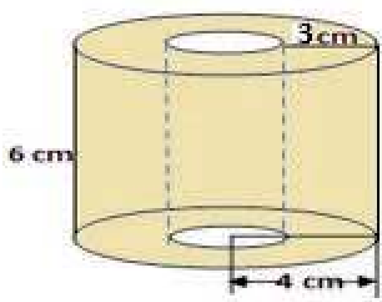



Figure 6.9

## Exercise 6.2

1. The radius of the base of a right circular cylinder is 4 cm and its altitude is 10 cm. Find the lateral surface area, the total surface area and the volume of the cylinder.
2. The diameter of the base of a right circular cylinder is 6 cm and its altitude is 8 cm. Find the lateral surface area, the total surface area and the volume of the cylinder.
3. A circular hole of radius 5 cm is drilled through the center of a right circular cylinder whose base has radius 7 cm and whose altitude is 8 cm. Find the total surface area and volume of the resulting solid.

## 6.2 Pyramids, Cones and Spheres

In your former grades, you learnt about the solid figures, pyramids and cones. Can you mention their shapes and names? You have learnt that based on the shape of their base; pyramids are classified into different types.

### Activity 6.3

1. Give answer for the following questions.

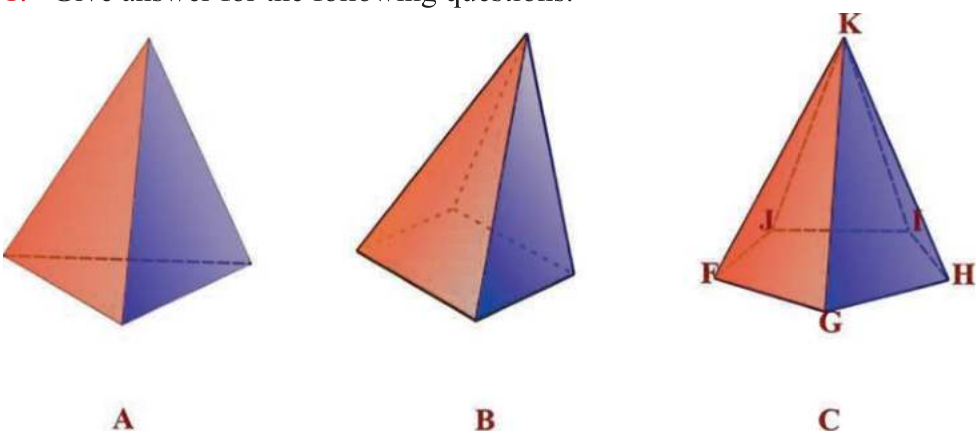



Figure 6.10

a. Name the pyramids in Figure 6.10

b. What is a tetrahedron?

c. What is a regular tetrahedron?

d. If the base of figure 6.10B is a rectangle, then the figure is called __________.

e. The region FGHIJ in figure 6.10C is called ____________ of the pyramid.

f. In Figure 6.10C the region KGH is called ___________________.

g. $\overline{KG}$ and $\overline{KH}$ are called ____________________.

h. $\overline{FG}$ and $\overline{GH}$ are called _______________.

2. What is altitude of a pyramid?

3. In Figure 6.10C, if FGHIJ were a circle then the solid would be called _______________.

4. Give two examples of solid figures that have the shape of a circular cone.

5. What is the shape of the lateral faces of a pyramid?

6. If the base of a pyramid is octagon, find the number of its lateral faces.

7. If the base of a pyramid is a decagon, find the number of its lateral faces.

8. In general, if the base of a pyramid has $n$ sides, find the number of its lateral faces.

## Definition 6.1

A pyramid is a solid figure defined by a polygonal base and a point called an apex (vertex) not on the base. It is formed when each point of the polygonal base is joined with the vertex.

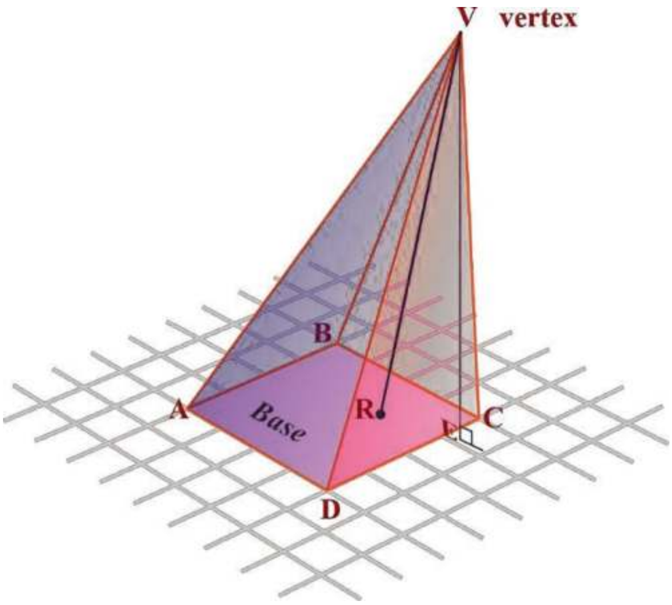



Figure 6.11

Figure 6.11 is a pyramid. The quadrilateral ABCD is the base and point V is the apex (vertex). For all points R on the base region, the collection of all line segments VR form the solid called a **pyramid**. The perpendicular line segment from the vertex to the plane containing the base is called the **height or altitude of a pyramid**. In Figure 6.11, VE is the height or altitude of the pyramid. If E is the center of the base region, then the pyramid is called **right pyramid,** that is, when the altitude meets the base at the center. Otherwise, it is called an **oblique pyramid**. The pyramid in figure 6.11 is oblique.

A regular pyramid is a pyramid whose base is a regular polygon and whose lateral edges are all equal in length.

Figure 6.12 is a regular pyramid, because its base is a square and the foot O of the altitude $\overline{NO}$ of the pyramid is the center of the base. The altitude NR of $\triangle$NKL is called **slant height of the Pyramid.**

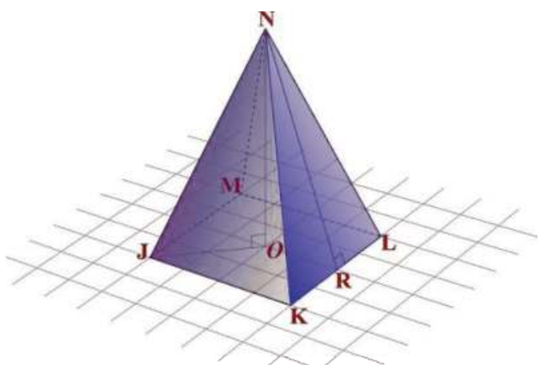



Figure 6.12

## Definition 6.2

A pyramid whose base is a circular region is called a circular cone.

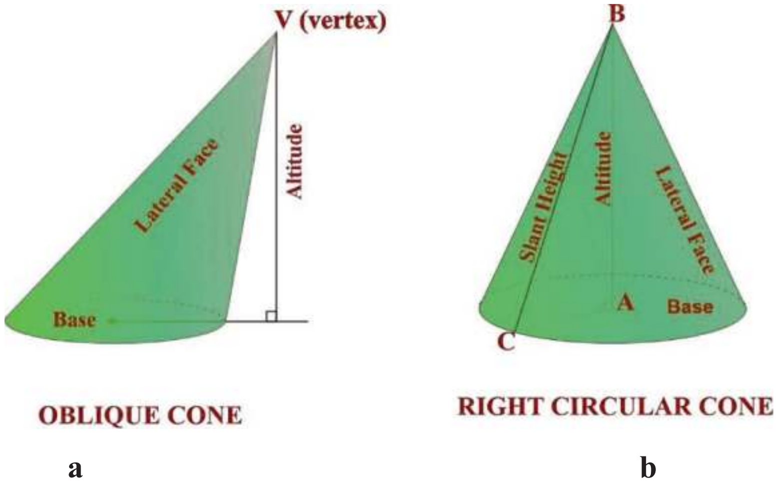



Figure 6.13

A right circular cone is a cone with a foot of its altitude at the center of the base circle. Figure 6.13b is a right circular cone while Figure 6.13a is an oblique cone. The line segment from the vertex to a point on the boundary of the base circle is called **the slant height of the cone**.

In figure 6.13b, $\overline{BC}$ is the slant height of the cone. The collection of all line segments BC where C is any point on the base circle makes the lateral surface of the cone (See figure 6.13b)

## Exercise 6.3

1. Name the following pyramids.

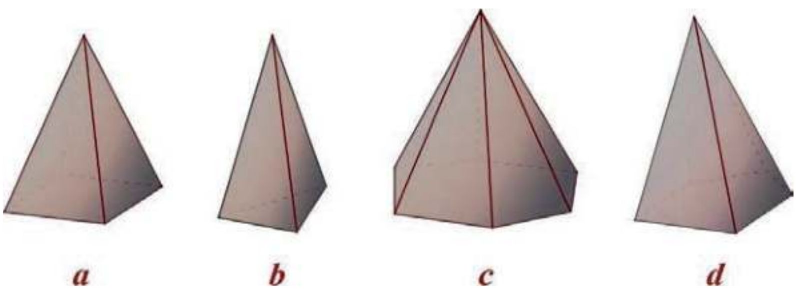



Figure 6.14

2. Figure 6.15 shows a circular cone whose base is a circle.
   a. What is PQ called?
   b. What is PR called?
   c. If PQ =4 cm and RQ=3 cm, find the length of PR.

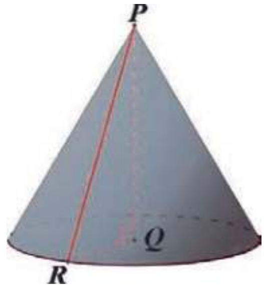



Figure 6.15

3. Determine whether each of the following statements is true or false.
   a. All lateral edges of a pyramid are equal in length.
   b. All lateral edges of a regular pyramid are equal in length.
   c. The length of the slant height of a right circular cone is greater than the length of its altitude.
   d. We can take any face of a triangular pyramid as its base.
   e. All faces of a regular pyramid are congruent.
   f. All lateral faces of a regular pyramid are congruent.

## 6.2.1 Surface Area of Pyramids and Cones

The surface area of any given object is **the area or region occupied by the surface of the object**, whereas volume is the amount of space available in an object. In geometry, there are different shapes and sizes such as sphere, cube, cuboid, cone, cylinder, etc. and each shape has its surface area as well as volume.

In general, the **lateral surface area** of a pyramid is the sum of the areas of its lateral faces and its **total surface area** is the sum of the areas of its lateral surface and its base.

If LSA denotes lateral surface area, TSA denotes total surface area and BA denotes base area then

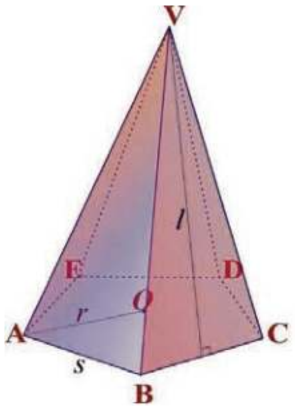
LSA = sum of areas of the lateral faces

TSA = LSA + BA

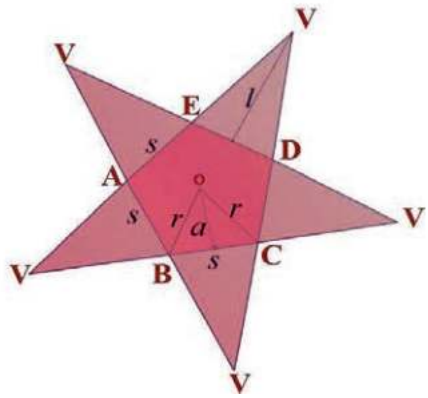
## Definition 6.3

A **geometry net** is a two-dimensional shape that can be folded to form a three-dimensional shape or a solid. When the surface of a three-dimensional figure is laid out flat showing each face of the figure, the pattern obtained is called **the net of the figure.**

To find the formula for the lateral surface area of a regular pyramid in terms of its **base perimeter** and **its slant height**, consider the regular pentagonal pyramid and its net.




a. Regular pentagonal Pyramid




b. Net of Figure a

Figure 6.16

In figure 6.16b, ABCDE is a regular pentagon and it is the base of the pyramid in figure 6.16a. Point O is the center, $a$ is the apothem, $s$ is the length of one side, $r$ is the radius of the base, ΔVED is one lateral face and $l$ is the slant height. All the five lateral faces are isosceles triangles and are congruent.

$$\text{area}(\triangle \text{VED}) = \frac{1}{2}sl$$

$$\text{LSA} = 5 \times \text{area}(\triangle \text{VED})$$

$$= 5(\frac{1}{2}sl) = \frac{1}{2}(5s)l$$

$$= \frac{1}{2}pl \text{ where } p = 5s \text{ is the perimeter of ABCDE.}$$

We can extend this idea for a regular polygon whose base has $n$ sides and generalize the surface area of a regular pyramid as follows.

The lateral surface area LSA of a regular pyramid is equal to half the product of its slant height and the perimeter of the base. That is,

$$\text{LSA} = \frac{1}{2}pl,$$

where $p = ns$ denotes the perimeter of the base, and $l$ denotes the slant height.

The total surface area TSA of a regular pyramid is given by

$$\text{TSA} = \text{BA} + \text{LSA} = \text{BA} + \frac{1}{2}pl$$

## Example 1

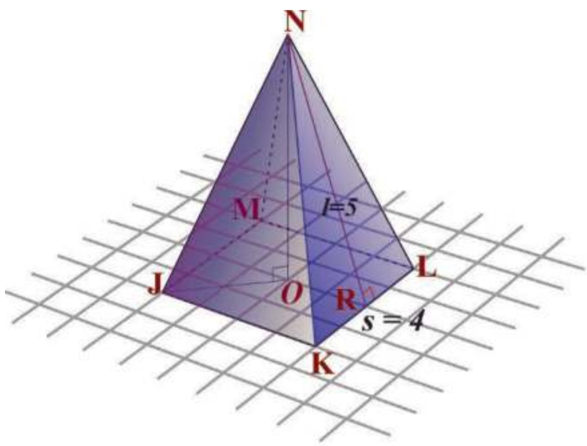
A regular square pyramid has a base edge of length 4 cm and slant height 5 cm.

**a.** Sketch the pyramid and its net.

**b.** Find its lateral and total surface area.

### Solution:

**a.**



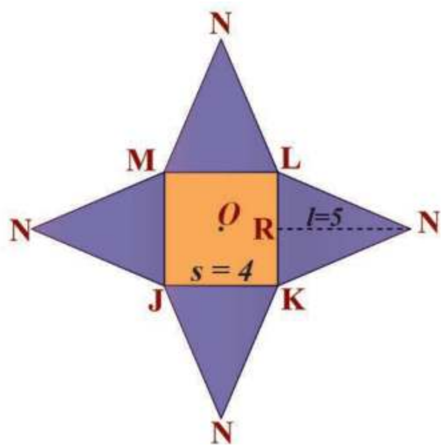




Figure 6.17

**b.** The base has 4 sides therefore, $n = 4$ and base perimeter

$p = ns = 4 \times 4 = 16.$

$\text{LSA} = \frac{1}{2}pl = \frac{1}{2} \times 16 \times 5 = 40 \text{ cm}^2.$

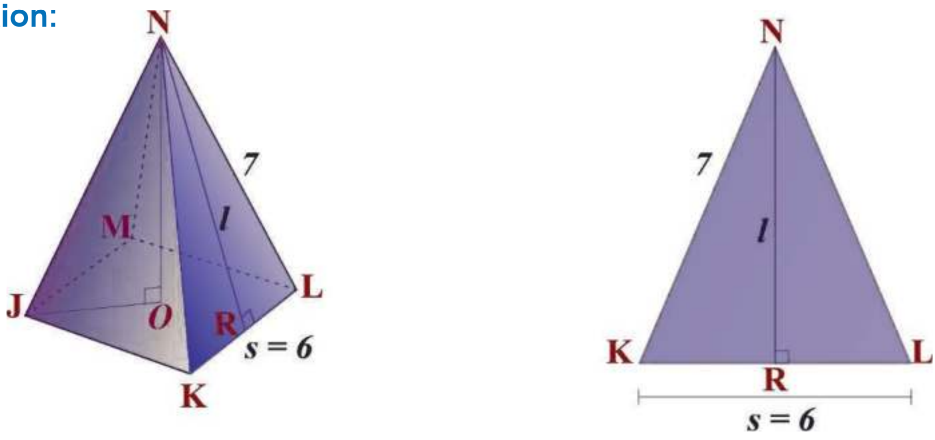
The base is a square. Therefore $BA = s^2 = 4^2 = 16\ \text{cm}^2$.

$TSA = LSA + BA = 40 + 16 = 56\ \text{cm}^2$.

### Example 2

A regular square pyramid has a base edge 6 cm and lateral edge 7 cm. Find its lateral and total surface areas.

**Solution:**




(a) (b)

Figure 6.18

ΔNKL in figure 6.18b is one lateral face of the pyramid in Figure 6.18a.

The lateral edge $\overline{NK} = 7$ cm, base edge $s = \overline{KL} = 6$ cm and $\overline{KR} = 3$ cm. The slant height $l = \overline{NR}$ of the pyramid is the altitude of ΔNKL.

Since ΔNRK is a right-angled triangle,

$$l^2 + (KR)^2 = (NK)^2$$
$$l^2 + 3^2 = 7^2$$
$$l^2 = 7^2 - 3^2 = 40$$
$$l = 2\sqrt{10}$$

$LSA = \frac{1}{2}pl = \frac{1}{2} \times 24 \times 2\sqrt{10} = 24\sqrt{10}\ \text{cm}^2$

$BA = s^2 = 6^2 = 36\ \text{cm}^2$

$TSA = LSA + BA = 24\sqrt{10} + 36 = 12(2\sqrt{10} + 3)\ \text{cm}^2$

## Example 3

A right pyramid with a rectangular base of length 4 cm and width 2 cm has altitude 3 cm.

**a.** Sketch the pyramid and its net.

**b.** Find its lateral and total surface areas.

### Solution:

**a.**

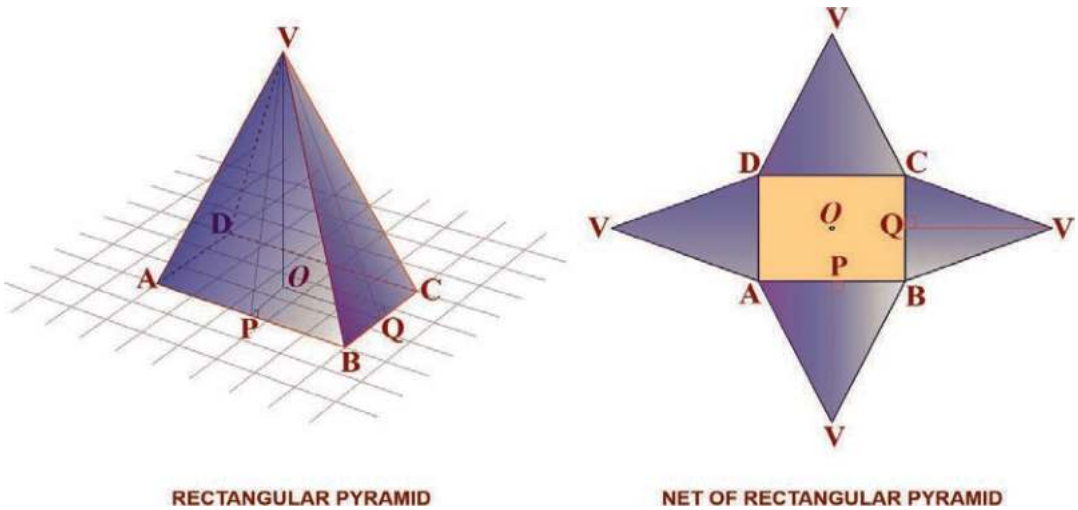



Figure 6.19

In figure 6.19, $h = \overline{VO}$ is the height (altitude) of the pyramid, ΔVBC, ΔVAD, ΔVAB and ΔVDC are lateral faces, $l_1 = \overline{VQ}$ and $l_2 = \overline{VP}$ are slant heights and they are altitudes of ΔVBC and ΔVAB respectively, and ABCD is the base of the pyramid.

**b.** $h = \overline{VO} = 3$ cm, $\overline{OQ} = 2$ cm *and* $\overline{OP} = 1$ cm.

$l_1 = \overline{VQ}$ and $(VO)^2 + (OQ)^2 = (VQ)^2$

$$3^2 + 2^2 = {l_1}^2$$

$${l_1}^2 = 13$$

$$l_1 = \sqrt{13}$$

$l_2 = \overline{VP}$ and $(VO)^2 + (OP)^2 = (VP)^2$

$$3^2 + 1^2 = {l_2}^2$$

$${l_2}^2 = 10$$

$$l_2 = \sqrt{10}$$

$\overline{BC} = \overline{AD} = w = 2$ cm and $\overline{AB} = \overline{DC} = l = 4$ cm, where $w$ and $l$ are width and length of the base rectangle ABCD

area of $\triangle$ VBC $= \frac{1}{2}bl_1 = \frac{1}{2} \times 2 \times \sqrt{13} = \sqrt{13}$

area of $\triangle$ VAB $= \frac{1}{2}ll_2 = \frac{1}{2} \times 4 \times \sqrt{10} = 2\sqrt{10}$

$\triangle$ VBC $\equiv \triangle$ VAD and $\triangle$ VAB $\equiv \triangle$ VAC

$$\text{LSA} = 2 \times \text{area}\,(\triangle \text{VBC}) + 2 \times \text{area}\,(\triangle \text{VAB})$$

$$= 2 \times \sqrt{13} + 2 \times 2\sqrt{10}$$

$$= 2(\sqrt{13} + 2\sqrt{10})\ \text{cm}^2,$$

$\text{BA} = b \times l = 2 \times 4 = 8\ \text{cm}^2$, and

$$\text{TSA} = \text{LSA} + \text{BA} = 2(\sqrt{13} + 2\sqrt{10})\ \text{cm}^2 + 8\ \text{cm}^2$$

$$= 2(\sqrt{13} + 2\sqrt{10} + 4)\ \text{cm}^2.$$

## Exercise 6.4

1. A regular square pyramid has a base edge of length 6 cm and slant height 8 cm.
   a. Sketch the pyramid and its net.
   b. Find its lateral and total surface area.
2. A right pyramid of 3 m height has a square base whose diagonal is 6 m. Find its lateral and total surface areas.
3. Find the lateral and the total surface areas of the regular hexagonal pyramid of 6 cm height and a length of 4 cm on one side of the base.

## Activity 6.4

1. A sector of a circle has radius $r$ and central angle $\theta$ as shown by Figure 6.20.
   a. Find the length of the arc.
   b. Find the area of the sector.
2. Find the arc length and area of the sector if $r = 6$ cm and $\theta = 30^0$.

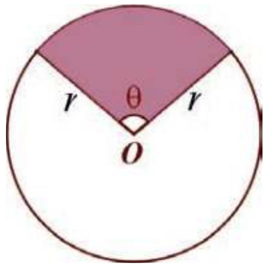



Figure 6.20

To find the lateral surface area of a right circular cone, cut a right circular cone (Figure 6.21a) along the slant height $l$, open up and flatten it. The resulting surface looks like a sector as shown in Figure 6.21c.

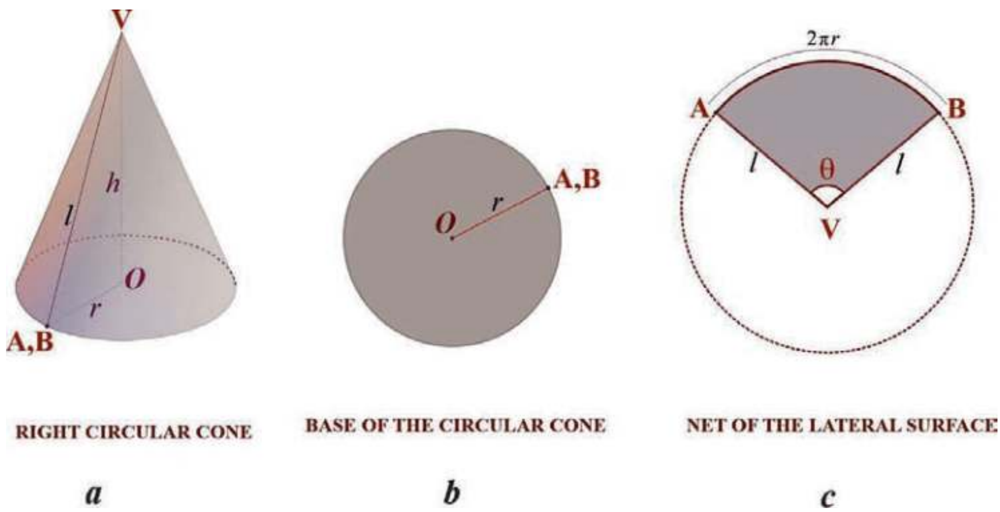



Figure 6.21

In figure 6.21c, length of arc$(AB) = \frac{\theta}{360}(2\pi l)$

Length of arc$(AB)$ = circumference of the base circle of figure 6.21a $= 2\pi r$

$$\frac{\theta}{360}(2\pi l) = 2\pi r$$

$$\frac{\theta}{360} l = r$$

LSA of the right circular cone = area of sector AVB $= \frac{\theta}{360}\pi l^2 = \pi\left(\frac{\theta}{360} l\right) l = \pi r l.$

LSA $= \pi r l = \frac{1}{2}(2\pi r)l = \frac{1}{2}pl$, where $p = 2\pi r$ is perimeter or circumference of the base circle.

The **lateral surface area** of a right circular cone is:

$$\text{LSA} = \pi r l \text{ or LSA} = \frac{1}{2}pl, \text{ where } p = 2\pi r$$

$l = \sqrt{h^2 + r^2}$ , where $l$ is the slant height, $h$ is the height (altitude), $r$ is the base radius and $p$ is perimeter or circumference.

The **total surface area** is the sum of the area of the base and the lateral surface area. That is,

$$\text{TSA} = \text{LSA} + \text{BA} = \pi r l + \pi r^2 = \pi r(l + r).$$

## Example 4

A right circular cone has a base diameter 8 cm and height 3 cm, see Figure 6.22. Find its LSA and TSA.

**Solution:**

$l = \sqrt{h^2 + r^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ cm,

$LSA = \pi r l = \pi(4)(5) = 20\pi$ cm$^2$,

$BA = \pi r^2 = 16\pi$ cm$^2$,

$TSA = LSA + BA = 36\pi$ cm$^2$.

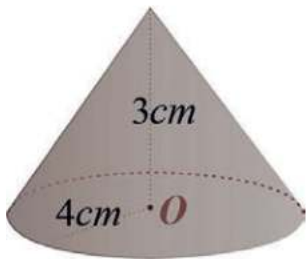



Figure 6.22

## Exercise 6.5

1. Find the total surface area of a right circular cone of radius 6 cm and height 12 cm.
2. The area of the total surface of a right circular cone is 64 m$^2$ and its slant height is 5 times the radius of the base. Find the radius of the base.

3. A conical tent is 6 m high and the radius of its base is 8 m. Find
   a. slant height of the tent.
   b. cost required to make the tent, if the cost of 1 $m^2$ canvas is Birr 250.
      Use $\pi = 3.14$

## 6.2.2 Horizontal cross-section of pyramids and cones

### Definition 6.4

If a pyramid or a cone is cut by a plane parallel to the plane containing the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal cross-section of the pyramid (or the cone).

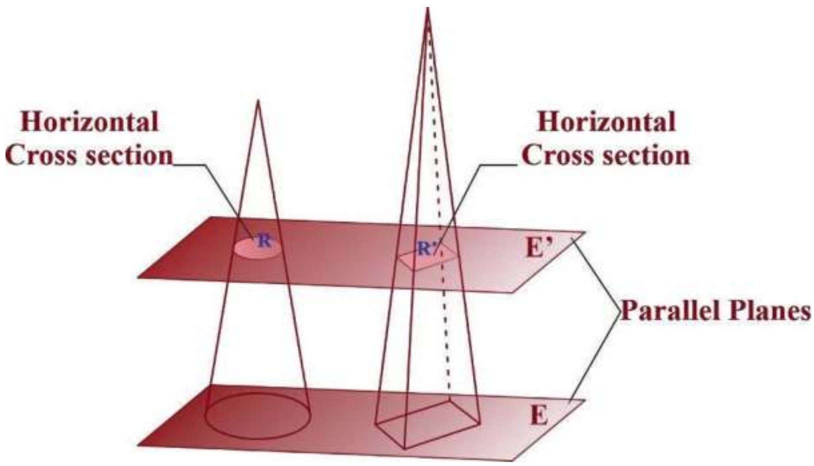



Figure 6.23

In figure 6.23, E is the plane containing the bases of the cone and the pyramid. E' is the plane parallel to E and intersects the cone and the pyramid along the regions R and $R'$, respectively. The regions R and $R'$ are called **horizontal cross-sections**.

### Theorem 6.1

Every horizontal cross-section of a triangular pyramid is a triangular region similar to the base.

## Proof:

Let the region of $\triangle ABC$ be the base of the pyramid lying in the plane E, $h$ be the altitude of the pyramid and $\triangle A'B'C'$ be the cross section at a distance $k$ from the vertex. Let D and $D'$ be the points in which the perpendicular line from V to E meets E and E', respectively.

To show that $\triangle A'B'C' \sim \triangle ABC$,

$\angle D'VA' \equiv \angle DVA$ (Common angles)

$\angle VD'A' \equiv \angle VDA \quad (\angle VD'A' = \angle VDA = 90^0)$

$\triangle VA'D' \sim \triangle VAD$ by AA similarity.

Thus, corresponding sides are proportional, that is,

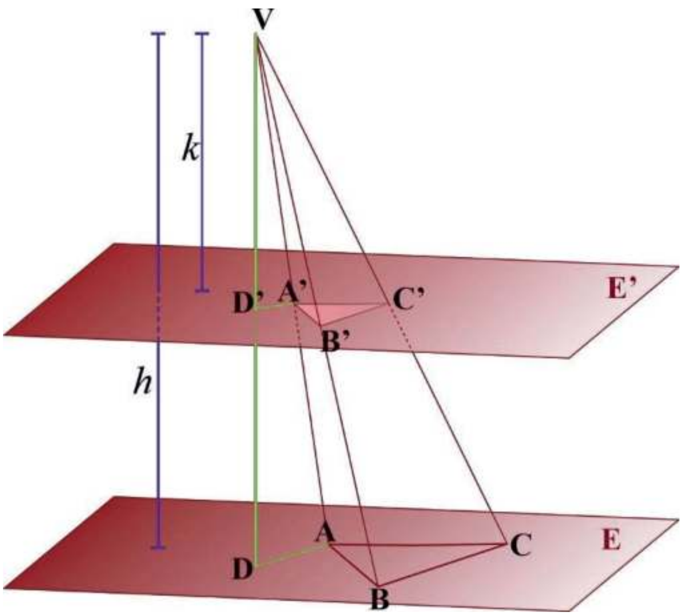



Figure 6.24

(1) $\frac{VA'}{VA} = \frac{VD'}{VD} = \frac{k}{h}$

Similarly, $\triangle VD'B' \sim \triangle VDB$ and hence

(2) $\frac{VB'}{VB} = \frac{VD'}{VD} = \frac{k}{h}$

(3) $\angle A'VB' = \angle AVB$ (Common angle)

Hence, from (1), (2) and (3) we have

$\triangle VA'B' \sim \triangle VAB$ by the SAS similarity.

Thus, corresponding sides are proportional, that is,

(4) $\frac{A'B'}{AB} = \frac{VA'}{VA} = \frac{k}{h}$

By using the same argument that leads to (4), we can also show that

$\Delta VB'C' \sim \Delta VBC$ to get (5)

(5) $\frac{B'C'}{BC} = \frac{k}{h}$

By showing $\Delta VA'C' \sim \Delta VAC$ we can get (6)

(6) $\frac{A'C'}{AC} = \frac{k}{h}$

Hence, from (4), (5) and (6) and by the *SSS* similarity theorem $\triangle A'B'C' \sim \triangle ABC$.

## Theorem 6.2

Let $h$ be the altitude of a triangular pyramid and let $k$ be the distance from the vertex to a horizontal cross-section. Then the ratio of the area of the cross section to the area of the base is $\frac{k^2}{h^2}$.

## Proof:

Let $\triangle ABC$ be the base, $h$ the altitude of the pyramid and $\triangle A'B'C'$ be the horizontal cross-section at a distance $k$ from the vertex (see Figure 6.24).

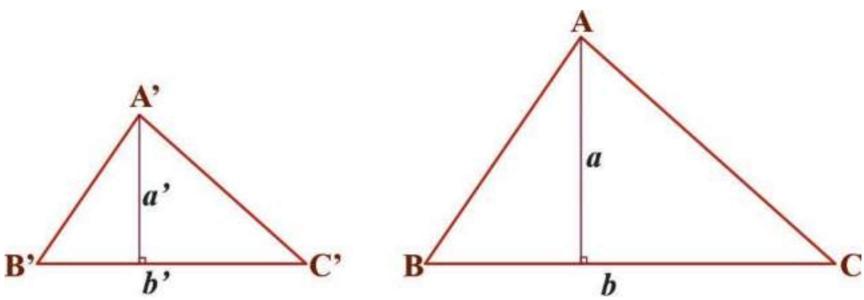



Figure 6.25

By theorem 6.1, $\triangle ABC \sim \triangle A'B'C'$

Note that the ratio between the altitudes of two similar triangles is the same as the ratio of their sides. So,

$$\frac{b'}{b} = \frac{a'}{a} \text{ --- } (1)$$

where $a'$ and $a$ are altitudes and $b$ and $b'$ are bases-in Figure 6.25.

On the other hand, in Figure 6.24,

$$\frac{\mathbf{B'C'}}{\mathbf{BC}} \left(= \frac{b'}{b}\right) = \frac{VC'}{VC} = \frac{VD'}{VD} = \frac{k}{h} \text{ --- } (2)$$

Hence,

$$\frac{\text{area of } \triangle A'B'C'}{\text{area of } \triangle ABC} = \frac{\frac{1}{2}(b'a')}{\frac{1}{2}(ba)} = \frac{b'}{b} \times \frac{a'}{a} = \frac{b'^2}{b^2} \qquad \text{(According to (1))}$$

$$= \frac{k^2}{h^2} \qquad \text{(According to (2))}$$

Thus, $\frac{\text{area of cross-section}}{\text{area of base}} = \frac{k^2}{h^2}$.

## Example 1

The area of the base of a triangular pyramid is 270 $cm^2$. The altitude of the pyramid is 6 cm. Find the area of the horizontal cross-section of the pyramid 4 cm from the vertex.

### Solution:

$h = 6\ cm, k = 4\ cm$ and base area of the pyramid $= 270\ cm^2$

$$\frac{\text{area of cross section}}{\text{base area}} = \frac{k^2}{h^2}$$

$$\frac{\text{area of cross section}}{270} = \frac{4^2}{6^2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Area of the cross section $= 270 \times \frac{4}{9} = 120\ cm^2$.

## Exercise 6.6

1. In the triangular pyramid shown in Figure 6.26, $\Delta A'B'C'$ is a horizontal cross-section. Find the values of $x$ and $\frac{k}{h}$.
2. The area of the base of a triangular pyramid is $100\ \text{cm}^2$. The altitude of the pyramid is 5 cm. Find the area of the horizontal cross-section of the pyramid 2 cm from the vertex.

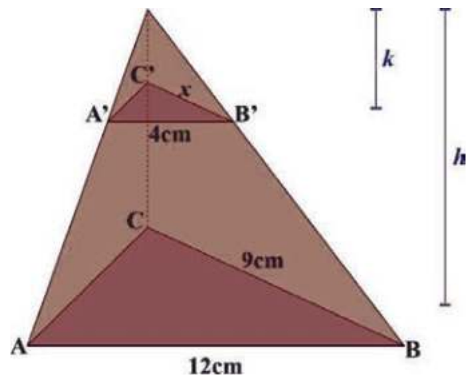
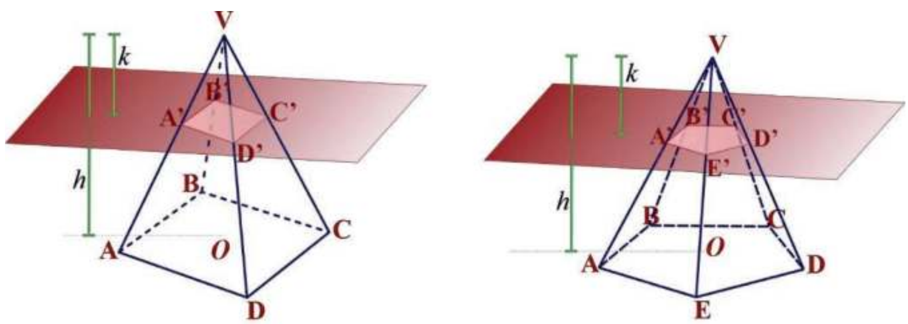


Figure 6.26

## Theorem 6.3

In any pyramid the ratio of a cross-section to the area of the base is $\frac{k^2}{h^2}$ where $h$ is the altitude of the pyramid and $k$ is the distance from the vertex to the plane of the cross section.




$$\frac{Area(A'B'C'D')}{Area(ABCD)} = \frac{k^2}{h^2} \qquad \frac{Area(A'B'C'D'E')}{Area(ABCDE)} = \frac{k^2}{h^2}$$

*a* *b*

Figure 6.27

## Example 2

The altitude of a square pyramid is 10 cm and a side of the base is 5 cm long. Find the area of the horizontal cross section of the pyramid 3 cm from the vertex.

**Solution:**

The base is a square of side $s = 5\ cm$, its area is $A = s^2 = 25\ \text{cm}^2$,

$$\frac{\text{area of cross section}}{\text{base area}} = \frac{k^2}{h^2}$$

$$\frac{\text{area of cross section}}{25} = \frac{3^2}{10^2}$$

$$\text{area of the cross section} = \frac{9}{4}\ \text{cm}^2$$

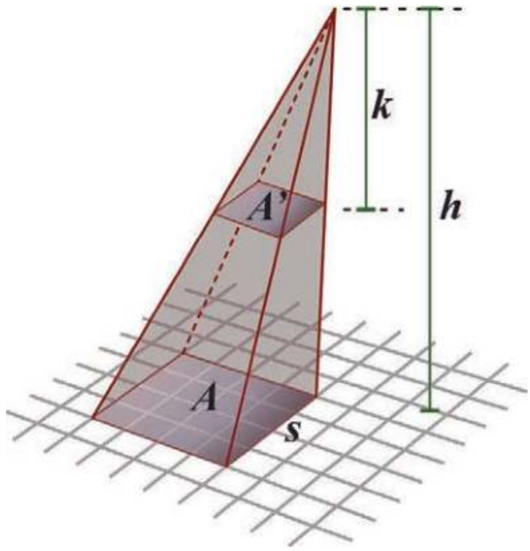



Figure 6.28

## Example 3

The area of the cross-section of a pyramid at a distance 5 cm from the base is $36\ \text{cm}^2$. If the area of the base of the pyramid is $81\ \text{cm}^2$, find the altitude of the pyramid.

**Solution:**

$\frac{\text{cross-section area}}{\text{base area}} = \frac{k^2}{h^2} = \frac{k^2}{(5+k)^2}$ and from this we have,

$\frac{36}{81} = \frac{k^2}{(5+k)^2}$ , where $k$ is the distance from the vertex of the pyramid to the cross section,

$\frac{6}{9} = \frac{k}{5+k}$ and solving for $k$ gives $k = 10$ cm.

Therefore, the altitude of the pyramid is 5 cm + 10 cm = 15 cm.

## Theorem 6.4

If two pyramids have the same base area and the same altitude then cross-sections equidistant from the vertices have the same area.

### Proof:

Let the two pyramids in figure 6.27 have the same base area $A$ and the same altitude $h$. Let $A'$ and $A''$ be the area of the cross sections $A'B'C'D'$ and $A'B'C'D'E'$, respectively. Let the distance from the vertices to the cross-sections be $k$.

By Theorem 6.3, $\frac{A'}{A} = \frac{k^2}{h^2} = \frac{A''}{A}$

$$\frac{A'}{A} = \frac{A''}{A}$$

$$A' = A''$$

## Exercise 6.7

1. The base of a pyramid is a rectangle with sides 6 cm and 4 cm. If the altitude of the pyramid is 12 cm, find the area of the horizontal cross-section of the pyramid 4 cm from the vertex.
2. The area of the horizontal cross-section of a pyramid at a distance 6 cm from the base is 90 $cm^2$. If the area of the base of the pyramid is 160 $cm^2$, find its altitude.
3. The radius of a horizontal cross-section of a cone at a distance 6 cm from the base is 2 cm. If the radius of the base of the cone is 3 cm, find its altitude.
4. The altitude of a regular hexagonal pyramid is 9 cm and the side of the base is 3 cm. What is the area of a horizontal cross-section at a distance of 5 cm from the base?
5. Prove Theorem 6.3

### 6.2.3 Volume of Pyramids and Cones

**CAVALIERI'S PRINCIPLE:** If two solids of equal height have equal cross-sectional areas at every level parallel to the respective bases, then the two solids have equal volume.

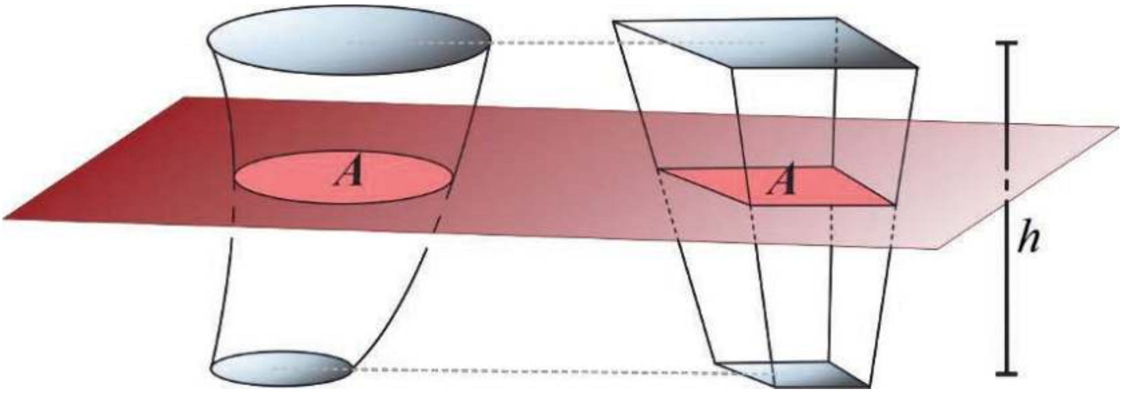



Figure 6.29

**Theorem 6.5**

If two pyramids have equal altitudes and equal base areas, then their volumes are equal.

### Proof:

By theorem 6.4, the two pyramids have equal cross-sectional area at every level parallel to the respective bases, then by the Cavalieri's Principle they have equal volume.

## Theorem 6.6

The volume of a triangular pyramid is one- third of the product of the height and the base area, that is, $V = \frac{1}{3}hB$.

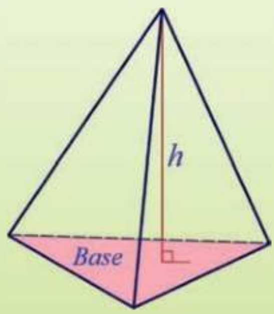



Figure 6.30

You know that the volume of a triangular prism is the product of its height and base area.

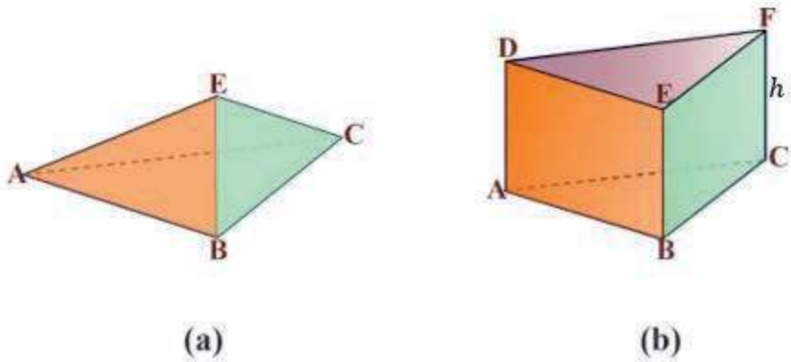



Figure 6.31

Figure 6.31b is a triangular prism. Let the height of the prism be $h$. Its volume is given by

$$V = (BA)h$$

where, BA is the base area, that is the area of $\Delta ABC$.

Let the triangular pyramid in figure 6.31a have the same base area and height as the prism.

We will show that the triangular prism is the union of three triangular pyramids, each having the same volume as the triangular pyramid in figure 6.31a. For this, we divide the triangular prism into three triangular pyramids as shown in figure 6.32.

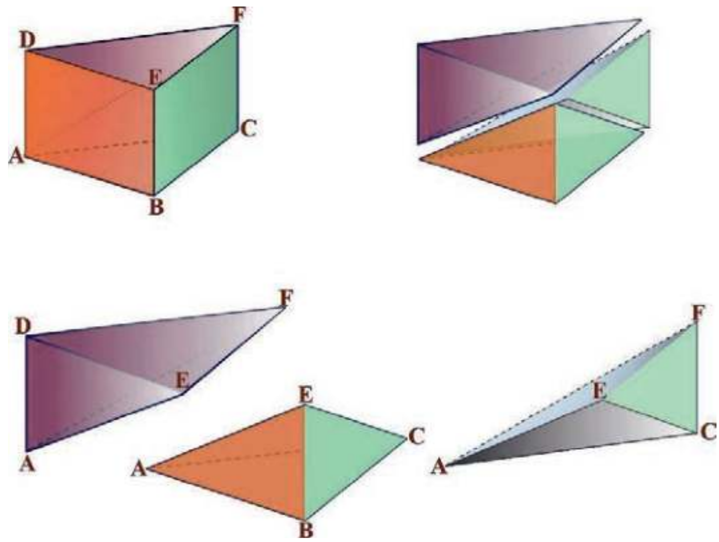



Figure 6.32

Consider triangular pyramids ADEF and ACEF with base ADF and ACF, respectively.

(1) AF is the diagonal of the parallelogram ADFC, $\triangle ADF \equiv \triangle ACF$.

(2) The pyramids have the same base area by (1).

(3) The pyramids have the same common vertex E.

(4) They have the same altitude. The altitude is the perpendicular distance from E to the parallelogram ADFC.

Hence by Theorem 6.5 they have the same volume. That is,

(5) Volume of the triangular pyramid ADEF = volume of triangular pyramid ACEF.

Consider the triangular pyramids ABCE and ACEF with base BCE and CFE respectively.

(6) CE is the diagonal of the parallelogram BCFE, $\Delta BCE \equiv \triangle FCE$.

(7) The triangles have the same base area by (6).

(8) The pyramids have the same common vertex A.

(9) They have the same altitude. The altitude is the perpendicular distance from A to the parallelogram BCFE.

Hence by Theorem 6.5 they have the same volume. That is,

(10) Volume of the triangular pyramid ABCE = Volume of the triangular pyramid ACEF.

(11) Volume of ABCE = Volume of ACEF = Volume of ADEF by (5) and (10).

Thus, all the three pyramids have the same volume. Since the volume of the prism is, $(BA)h$, the volume of each of the pyramid becomes $\frac{1}{3}(BA)h$.

Therefore, the volume of the pyramid in figure 6.31a. is $\frac{1}{3}(BA)h$ and this proves Theorem 6.6.

## Theorem 6.7

The volume of a pyramid is one third of the product of its altitude and its base area.

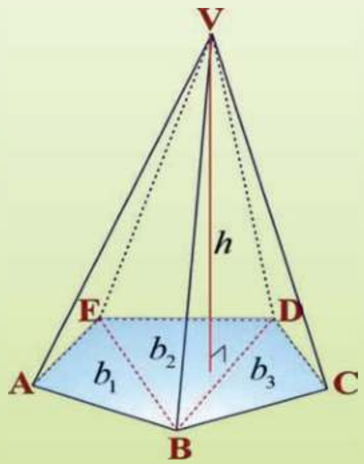



Figure 6.33

## Proof:

Consider any pyramid with base area B and altitude $h$. Since the base is a polygonal region, it can be subdivided into a finite number of triangular regions with areas

$b_1, b_2, \ldots, b_n$. Hence the volume of the given pyramid is the sum of the volumes of triangular pyramids which have the same altitude $h$.

Hence the volume is,

$$V = \frac{1}{3}b_1h + \frac{1}{3}b_2h + \ldots + \frac{1}{3}b_nh$$
$$= \frac{1}{3}(b_1 + b_2 + \ldots + b_n)h = \frac{1}{3}(\text{BA})h$$

## Theorem 6.8

The volume V of a circular cone with altitude $h$ and base radius $r$ is,

$$V = \frac{1}{3}\pi r^2 h.$$

### Example 1

A regular square pyramid has a base edge of length 4 cm and altitude 7 cm. Find its volume.

**Solution:**

The base of the pyramid is a square with base edge of length 4 cm.

$\text{BA} = 4 \times 4 = 16 \text{ cm}^2$

$V = \frac{1}{3}(\text{BA})h = \frac{1}{3} \times 16 \times 7$

$= \frac{112}{3} \text{ cm}^3.$

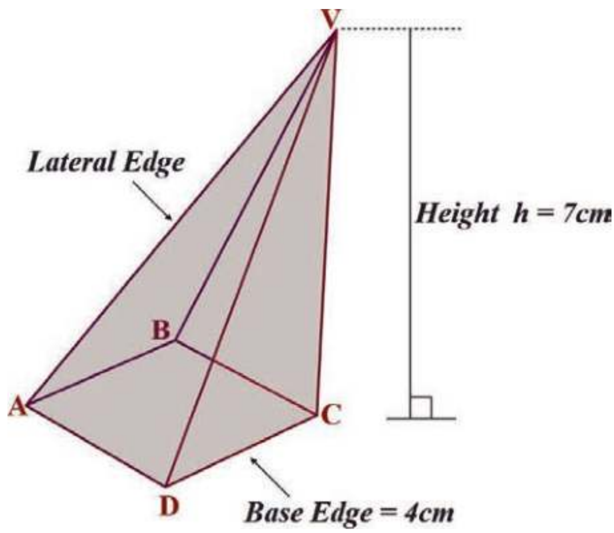



Figure 6.34

### Example 2

A circular cone has a base radius 5 cm and altitude 6 cm. What is its volume?

**Solution:**

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(5^2)(6)$$

$$= 50\pi \text{ cm}^3$$

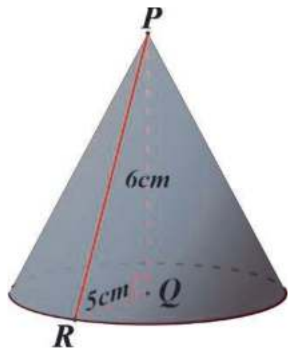



Figure 6.35

## Exercise 6.8

1. The altitude of a regular square pyramid is 6 cm. If one edge of the base has length 4 cm then find its volume.
2. A circular cone has an altitude 12 cm and a base radius 10 cm. What is its volume?
3. A right circular cone has height 10 cm and circumference of the base is $12\pi$ cm. Find its volume.
4. The lateral edge of a regular tetrahedron is 6 cm. Find its total surface area and its volume.
5. A right circular conical vessel of altitude 20 cm and base radius 10 cm is kept with its vertex downwards. If one liter of water is poured into it, how high above the vertex will the level of the water be? Use $\pi = 3.14$

## 6.2.4 Surface Area and Volume of Sphere

**Definition 6.5**

A sphere is a solid bounded by a closed surface every point of which is equidistant from a fixed point called the center.

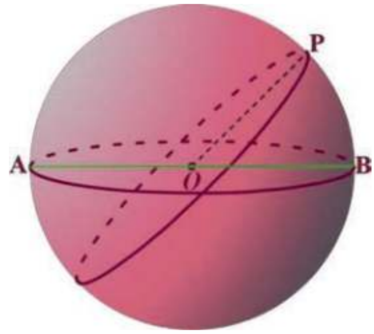


Figure 6.36

**Radius** of a sphere is a line segment connecting its center with any point on the sphere.

**Diameter of the sphere** is a line segment from the surface of the sphere passing through the center and ending at the surface.

In Figure 6.36, O is the center, OP is the radius and AB is the diameter of the sphere. Moreover, OA and OB are also the radius of the sphere.

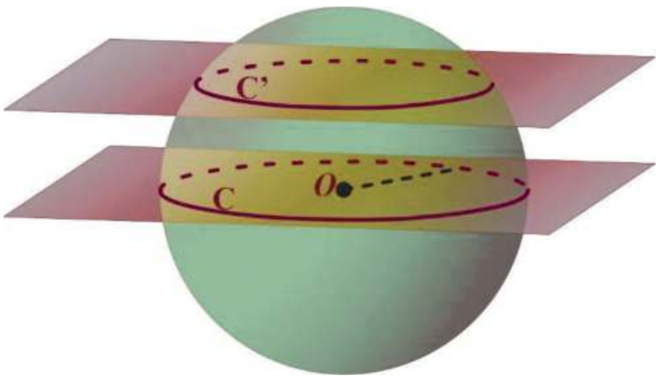


Figure 6.37

## Great and Small Circles

Every cross-section made by a plane passed through a sphere is a circle.

If the plane passes through the center of a sphere, the cross-section formed is **a great circle**; otherwise, the cross-section is **a small circle**. Clearly any plane through the center of the sphere contains a diameter. Hence all great circles of a sphere are equal and have for their common center, the center of the sphere and have for their radius, the radius of the sphere. In Figure 6.37, C′ is a small circle and C is a great circle of the sphere.

## Hemisphere

A great circle bisects the surface of a sphere. One of the two equal parts into which the sphere is divided by a great circle is called a **hemisphere**. Figure 6.38 is a hemisphere.

Figure 6.38

### Activity 6.5

1. Give examples of objects from your surroundings that have spherical shapes.
2. Give two examples of hemispheres.

If $r$ is the radius, SA surface area and V volume of a sphere, then

$$SA = 4\pi r^2,$$

$$V = \frac{4}{3}\pi r^3.$$

### Exercise 6.9

1. The radius of a sphere is 10 cm. Find its surface area and volume.
2. The diameter of an iron ball is 6 cm. Find its surface area and volume (use $\pi = 3.14$).
3. Find the formula for the surface area and volume of a sphere in terms of its diameter $d$.

## 6.3 Frustum of Pyramids and Cones

### Activity 6.6

1. What is a trapezium?
2. Find the area of a trapezium whose parallel sides are 10 cm and 14 cm and the distance between them is 8 cm.
3. The area of a trapezium is 150 $cm^2$ and the distance between its parallel sides is 16 cm. If one of the parallel sides is 8 cm, find the other.
4. In Figure 6.39, sector of two concentric circles of radii 9 cm and 11 cm are shown. Find the area of the shaded region in terms of $\pi$.

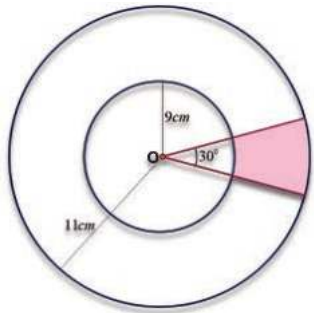



Figure 6.39

### Definition 6.6

A frustum of a pyramid is part of the pyramid between the base and the vertex formed when the original pyramid is cut off by a plane parallel to the plane of the base. That is, the frustum of a pyramid is part of the pyramid between the base and a cross-section of the pyramid.

When a pyramid is cut by a plane parallel to the base, the part of the pyramid between the vertex and the cross-section is again a pyramid whereas the other part is not a pyramid.

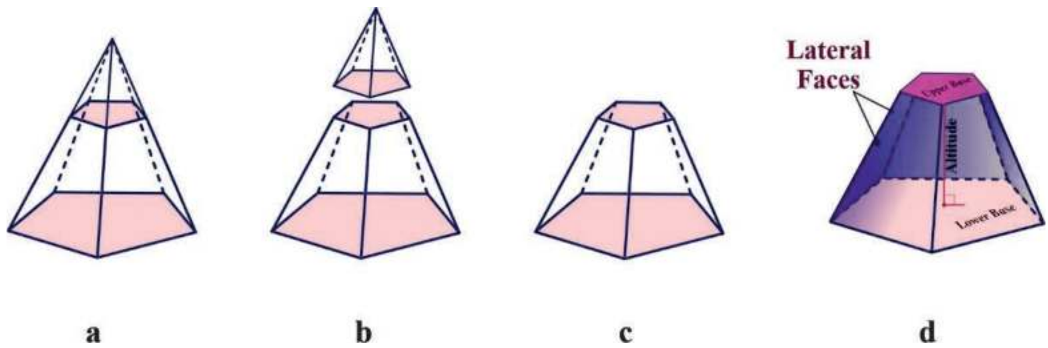



Figure 6.40

Figure 6.40c is the frustum of a pyramid. The base of the pyramid and the cross-section are called the bases of the frustum. The other faces are called lateral faces. The lateral surface of the frustum is the sum of the lateral faces. The total surface is the sum of the lateral surface and the bases.

The altitude of a frustum of a pyramid is the perpendicular distance between the bases.

**Observations:**

1. The lateral faces of a frustum of a pyramid are trapezium.
2. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.
3. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.
4. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral faces.

**Definition 6.7**

A frustum of a cone is a part of the cone between the base and the vertex formed when the original cone is cut off by a plane parallel to the plane of the base.

The slant height of a frustum of a right circular cone is the part of the slant height of the cone which is included between the bases.

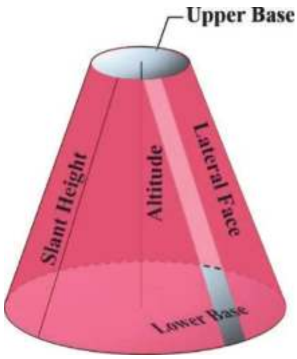



Figure 6.41

## Activity 6.7

Give at least two examples of objects from your surroundings that have the shape of frustum of a right circular cone.

## Example 1

The lower base of the frustum of a regular pyramid is a square of side of length $s$ unit and the upper base has a side of length $s'$ unit. If the slant height is $l$ unit long then find,

**a.** the lateral surface area of the frustum.

**b.** the total surface area of the frustum.

**c.** Find the values of the lateral surface area and the total surface area, when $s = 5$ cm, $s' = 3$ cm and $l = 4$ cm.

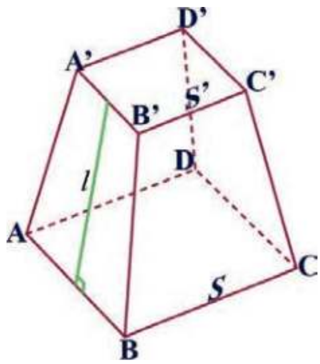



Figure 6.42

**Solution:**

The four faces are congruent isosceles trapeziums.

**a.** LSA = $\text{area}(A'ABB') + \text{area}(B'BCC') + \text{area}(C'CDD') + \text{area}(D'DAA')$

$= 4 \times \frac{1}{2} l(s + s')$

$= \frac{1}{2} l(4s + 4s')$ ( Note here that $4s$ *and* $4s'$ are the perimeters of the lower and the upper bases, respectively)

$= 2l(s + s')$.

**b.** $TSA = LSA + BA = 2l(s + s') + s^2 + s'^2$.

**c.** $LSA = 2(4)(5 + 3) = 64\ cm^2$, and

$TSA = 64 + (5^2 + 3^2) = 98\ cm^2$.

The lateral surface area (LSA) of a frustum of a regular pyramid is equal to half the product of the slant height $l$ and the sum of the perimeter $p$ of the lower base and perimeter $p'$ of the upper base. That is, $LSA = \frac{1}{2}l(p + p')$.

## Exercise 6.10

The lower base of the frustum of a regular pyramid is an equilateral triangle of side of length 8 cm and the upper base has a side of length 4 cm. If the slant height is 5 cm, then find

a. the lateral surface area of the frustum

b. the total surface area of the frustum

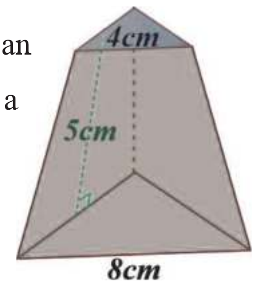



Figure 6.43

## Activity 6.8

Consider figure 6.44 and find

a. LSA of the bigger cone.

b. LSA of the smaller cone.

c. LSA of the frustum.

d. The volume of the bigger cone.

e. The volume of the smaller cone.

f. The volume of the frustum.

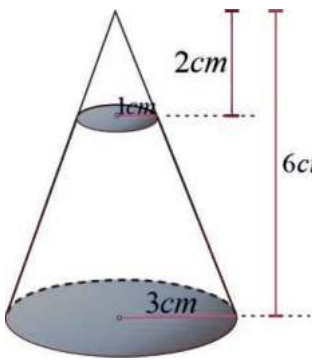



Figure 6.44

### Example 2

From a right circular cone of altitude 8 cm and base radius 6 cm a frustum of height 4 cm is formed. What is the lateral surface area of the frustum?

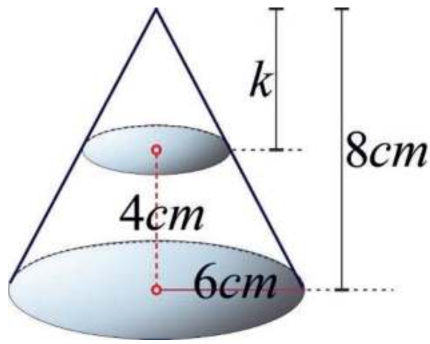



Figure 6.45

**Solution:**

In Figure 6.45, let CA denote the cross-section area, $k$ and $h$ are altitudes of the smaller cone and the bigger cone, respectively. Assume also that $l'$ and $l$ are slant heights of the smaller and the bigger cones, respectively. Then,

$$\frac{\text{area of the cross-section}}{\text{base area}} = \frac{\text{CA}}{\text{BA}} = \left(\frac{k}{h}\right)^2, \; k = h - 4 = 4 \text{ cm}$$

$\frac{CA}{36\pi} = \frac{1}{4}$, since BA = $\pi r^2 = 36\pi$

CA = $\frac{36\pi}{4} = 9\pi \text{ cm}^2$ ... (*)

CA = $\pi(r')^2$, where $r'$ is the radius of the cross section

$9\pi = \pi(r')^2$, using CA = $9\pi$ from (*)

$r' = 3$ cm

$l = \sqrt{r^2 + h^2} = \sqrt{6^2 + 8^2} = 10$ cm and

$l' = \sqrt{r'^2 + k^2} = \sqrt{3^2 + 4^2} = 5$ cm

LSA of the smaller cone = $\pi r' l' = 3 \times 5 \times \pi = 15\pi \text{ cm}^2$.

LSA of the bigger cone = $\pi r l = 6 \times 10 \times \pi = 60\pi \text{ cm}^2$.

Hence, lateral surface area of the frustum

= (LSA of the bigger cone) - (LSA the smaller cone)

= $60\pi \text{ cm}^2 - 15\pi \text{ cm}^2 = 45\pi \text{ cm}^2$.

## Exercise 6.11

For the right circular cone in Figure 6.46,

a. Find the height of the smaller cone.

b. Find the Lateral Surface Area (LSA) of the frustum using

LSA = (LSA of the larger cone)

– (LSA of the smaller cone).

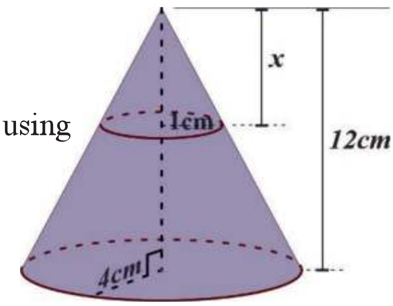



Figure 6.46

To find the formula for the lateral surface area of a frustum of a right circular cone, in terms of its base radii $r$ and $r'$ and its slant height $l$, let us consider a right circular cone and its net as shown by Figure 6.47.

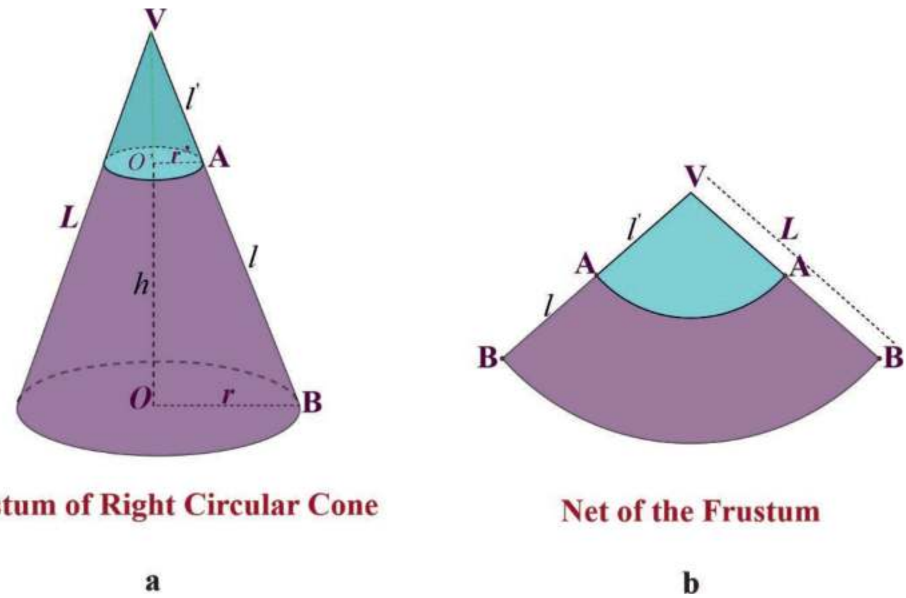



Figure 6.47

From Figure 6.47a, as $\triangle VO'A \sim \triangle VOB$ we have

$$\frac{l'}{L} = \frac{r'}{r} \quad . \; . \; . \quad (1)$$

$$l' = L - l \quad . \; . \; . \quad (2)$$

Substituting the value of $l'$ of (2) to the value of $l'$ in (1) and solving for L gives,

$L = l\left(\frac{r}{r-r'}\right) \quad . \; . \; . (3)$

Frustum area = (area of the bigger cone) - (area of the smaller cone)

$= \pi r L - \pi r' l' \quad . \; . \; . \; (4)$

$= \pi r L - \pi r'(L - l)$ (Substituting the values of $l'$ from equation (2) to equation (4))

$= \pi (r - r')L + \pi r' l \quad . \; . \; . \; (5)$

$= \pi (r - r') l \left(\frac{r}{r-r'}\right) + \pi r' l = \pi r l + \pi r' l$ (Substituting the values of $L$ from equation (3) to equation (5))

$= \pi l (r + r').$

> For a frustum of a right circular cone with slant height $l$, if the radii of the bases are $r$ and $r'$, then the lateral surface area of the frustum is given by
>
> $\text{LSA} = l\pi(r + r').$

## Example 3

Find the lateral and total surface area of the frustum of a right circular cone of height 24 cm, base radius 22 cm, and slant height 26 cm. Use $\pi = 3.14$.

**Solution:**

Let the larger base radius of the frustum be $r = 22$ cm and assume that its small radius is $r'$.

The height of the frustum of the cone is $h = 24$ cm and its slant height is $l = 26$ cm.

Also consider Figure 6.48,

$r = \overline{\text{OB}}, h = \overline{\text{PO}} = \overline{\text{AR}}, r' = \overline{\text{PA}}$ *and* $l = \overline{\text{AB}}$.

OPAR is a rectangle and $\overline{\text{PA}} = \overline{\text{OR}} = r'$.

$\Delta$ARB is right-angled triangle and $\overline{\text{AB}}^2 = \overline{\text{AR}}^2 + \overline{\text{RB}}^2$

$\overline{\text{AB}}^2 = \overline{\text{AR}}^2 + (\overline{OB} - \overline{OR})^2$

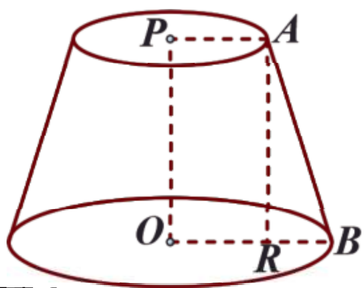



Figure 6.48

$l^2 = h^2 + (r - r')^2$

$26^2 = 24^2 + (22 - r')^2$, solving for $r'$ from this equation $r' = 12$ cm.

LSA $= l\pi(r + r') = 26\pi(22 + 12) = 884\pi$ cm$^2$.

BA = the area of the two bases $= \pi(r^2 + r'^2) = \pi(22^2 + 12^2) = 628\pi$ cm$^2$.

TSA = LSA + BA $= 884\pi + 628\pi = 1512\pi$ cm$^2$

$= 1512 \times 3.14 = 4747.68$ cm$^2$

## Exercise 6.12

1. Find the lateral and total surface area of the frustum of a right circular cone of height 9 cm and base radii 4 cm and 1 cm.
2. A right circular cone is cut by a plane parallel to the base at a distance 9 cm from the vertex to form a frustum of a cone. If the radii of the bases of the frustum formed are 3 cm and 5 cm, then find the altitude and the lateral surface area of the frustum.

To find the volume formula for a frustum of a right circular cone in terms of its base radius $r$ and $r'$ and its height $h$, consider Figure 6.49.

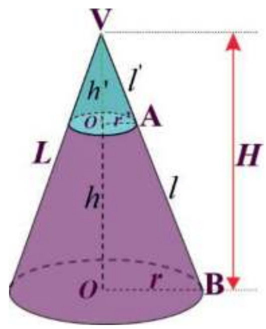



Figure 6.49

$$\text{H} = h + h' \quad . \ . \ . \quad (1)$$

$$\frac{h'}{\text{H}} = \frac{r'}{r} \quad . \ . \ . \quad (2)$$

Substituting the value of H in (1) to the value of H in (2)

$$\frac{h'}{h + h'} = \frac{r'}{r} \quad . \ . \ . \quad (3)$$

Solving for $h'$

$$h' = \frac{hr'}{r - r'} \quad . \ . \ . \quad (4)$$

Volume of the frustum = (volume of the bigger cone) - (volume of the smaller cone)

$$= \tfrac{1}{3}\pi r^2\text{H} - \tfrac{1}{3}\pi r'^2 h' = \tfrac{1}{3}\pi r^2(h + h') - \tfrac{1}{3}\pi r'^2 h'$$

$$= \frac{1}{3}\pi r^2 h + \frac{1}{3}\pi r^2 h' - \frac{1}{3}\pi r'^2 h'$$

$$= \frac{1}{3}\ \pi(r^2 h + (r^2 - r'^2)h')$$

$$= \frac{1}{3}\pi\ (r^2 h + \frac{(r^2 - r'^2)hr'}{r - r'}) \quad \text{by (4)}$$

$$= \frac{1}{3}\pi\ (r^2 h + \frac{(r + r')(r - r')hr'}{r - r'})$$

$$= \frac{1}{3}\pi(r^2 h + (r + r')hr')$$

$$= \frac{1}{3}\pi h(r^2 + r'^2 + rr')$$

The volume of a frustum of a right circular cone is

$$V = \frac{1}{3}\pi h(r^2 + r'^2 + rr'),$$

where $r$ is the radius of the bigger circle, $r'$ is the radius of the smaller circle and $h$ is the altitude of the frustum.

## Example 4

Find the volume of the frustum of a cone whose top and bottom diameters are 6 m and 10 m and the height is 12 m. (Use $\pi$=3.14)

### Solution:

Since the diameter of the upper base is 6 m the radius of the upper base becomes 3 m and since the diameter of the lower base is 10 m, its radius becomes 5 m. The height is 12 m, so

$$\text{Volume } = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$$

$$= \frac{1}{3}\pi(12)(3^2 + 5^2 + (3)(5))$$

$$= 196\pi = 196 \times 3.14 = 615.44 \text{ m}^3.$$

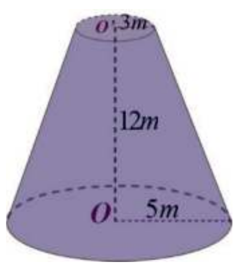
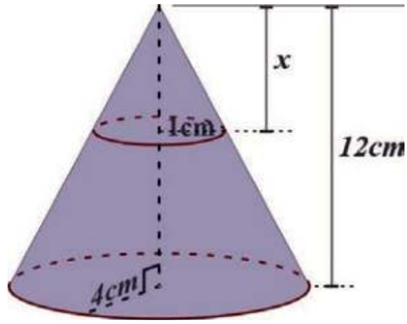



Figure 6.50

## Exercise 6.13

1. Given the same right circular cone as in Figure 6.46 above,
   a. Calculate the volume of the larger cone.
   b. Calculate the volume of the smaller cone.
   c. Find the volume of the frustum by V = (volume of the larger cone) – (volume of the smaller cone).
   d. Find the volume by using the formula, $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$ and compare the results.




(Fig. 6.46 reproduced)

2. A right circular cone of altitude 16 m is cut by a plane parallel to the base 4 m from the vertex to form a frustum of the cone. If the radius of the base of the right circular cone is 12 m, then find
   a. the radius of the other base of the frustum.
   b. the volume of the frustum?

The volume formula, $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$, can be rewritten as

$$= \frac{1}{3}h\left(\pi r^2 + \pi r'^2 + \sqrt{(\pi r^2)(\pi r'^2)}\right)$$

$$= \frac{1}{3}h(A + A' + \sqrt{AA'})$$

where A and A′ are base areas and *h* is the altitude of the frustum.

The volume of a frustum of a right circular cone is

$$V = \frac{1}{3}h(A + A' + \sqrt{AA'}),$$

where A and A′ are base areas and *h* is the altitude of the cone.

The volume of a frustum of a pyramid is also, $V = \frac{1}{3}h(A + A' + \sqrt{AA'})$, where A and A′ are base areas and *h* is the altitude of the frustum.

## Exercise 6.14

1. A right circular cone of altitude 9 cm is cut by a plane parallel to the base 7 cm from the vertex. If the area of the base of the frustum formed are 81 $cm^2$ and 49 $cm^2$ then find the volume of the frustum.
2. The lower base of a frustum of a regular pyramid is a square of side of length 6 cm, and the upper base has side length 4 cm. If the slant height is 8 cm, find

   a. its lateral surface area  b. its total surface area  c. its volume
3. What is the lateral area of a regular pyramid whose base is a square 12 cm on a side and whose slant height is 10 cm? If a plane is passed parallel to the base and 4 cm from the vertex, what is the lateral surface area and volume of the frustum?
4. A frustum of a regular square pyramid has a height of 2 cm. The lateral faces of the pyramid are equilateral triangles of side $3\sqrt{2}$ cm. Find the volume of the frustum.
5. A cone 12 cm high is cut 8 cm from the vertex to form a frustum with a volume of 156 $cm^3$. Find the radius of the bases of the cone.
6. Show that the volume of a frustum of a pyramid is $V = \frac{1}{3}h(A + A' + \sqrt{AA'})$, where A and A′ are base areas and $h$ is the altitude of the frustum.

## 6.4 Surface Area and Volume of Composed Solids

A composed solid is a solid that is made up of two or more solids. In order to find the volume and surface area of a composed solid, one needs to identify the different parts it is made of. This decomposition allows working out the volume and surface area of each part independently. The volume of the composed solid is simply the sum of the volumes of its parts.

To find the surface area and volume of a composed solid, you need to know how to find the surface area and volume of prisms, pyramids, cones, cylinders, and spheres.

In the preceding sections you learnt how to calculate the surface area and volumes of these solids.

### Example 1

Find the total surface area and volume of the flask shown in figure 6.51.

**Solution:**

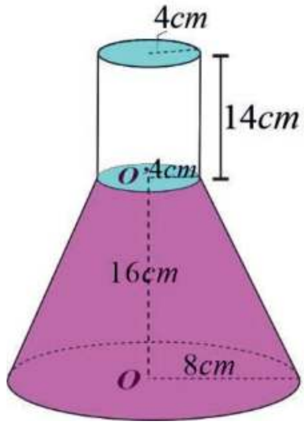



Figure 6.51

The total surface area of the flask is the lateral surface area of the frustum of the cone, lower base area of the frustum of the cone and lateral surface area of the cylinder.

Slant height $l$ of the frustum of the cone is:

$$l = \sqrt{4^2 + 16^2} = \sqrt{272} = 4\sqrt{17} \text{ cm},$$

Lateral surface area of the frustum $= l\pi(r + r')$

$$= 48\sqrt{17}\pi \text{ cm}^2,$$

Lower base area of the frustum $= \pi r^2 = 64\pi \text{ cm}^2$,

Lateral surface area of the cylinder $= 2\pi rh = 112\pi \text{ cm}^2$.

Total surface area $= 48\sqrt{17}\pi \text{ cm}^2 + 64\pi \text{ cm}^2 + 112\pi \text{ cm}^2 = 16\pi(3\sqrt{17} + 11) \text{ cm}^2$.

Total volume of the flask = Volume of the frustum of the cone + volume of the cylinder

$$= \frac{\pi}{3}h'(r^2 + r'^2 + rr') + \pi r^2 h$$

$$= \frac{16\pi}{3}(8^2 + 4^2 + 8(4)) + \pi 8^2(14)$$

$$= \frac{1792}{3}\pi + 896\pi = \frac{4480}{3}\pi \text{ cm}^3.$$

### Example 2

A solid figure is made of a circular cylinder of radius 4 cm at bottom and a right circular cone of altitude 3 cm, as shown in Figure 6.52. If the overall height is 10 cm, find the total surface area and the volume of the solid.

**Solution:**

Figure 6.52 is a composed solid because it is made up of two solids, a cone and a cylinder. The total surface area of the solid is the sum of the area of three surfaces. These are the lateral surface of the cone, the lateral surface of the cylinder and the lower base of the cylinder.

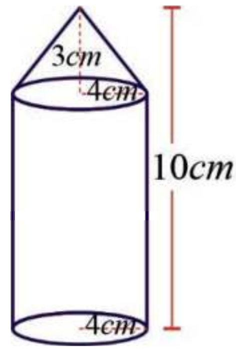



Figure 6.52

Slant height $l$ of the cone

$$l = \sqrt{h^2 + r^2} = \sqrt{4^2 + 3^2} = 5 \text{ cm},$$

LSA of the cone = $\pi r l = 4 \times 5 \times \pi = 20\pi \text{ cm}^2$ ,

Height of the cylinder is 7cm,

LSA of the cylinder = $2\pi r h = 2\pi \times 4 \times 7 = 56\pi \text{ cm}^2$,

Lower base area of the cylinder = $\pi r^2 = 16\pi \text{ cm}^2$,

Total surface area of the solid = $20\pi + 56\pi + 16\pi = 92\pi \text{ cm}^2$.

Total volume of the solid = Volume of the cylinder + Volume of the cone

$$= \pi r^2 h + \frac{1}{3}\pi r^2 h = \pi(4^2)(7) + \frac{1}{3}\pi(4^2)(3) = 128\pi \text{ cm}^3.$$

## Exercise 6.15

1. Find the total surface area and volume of the following.

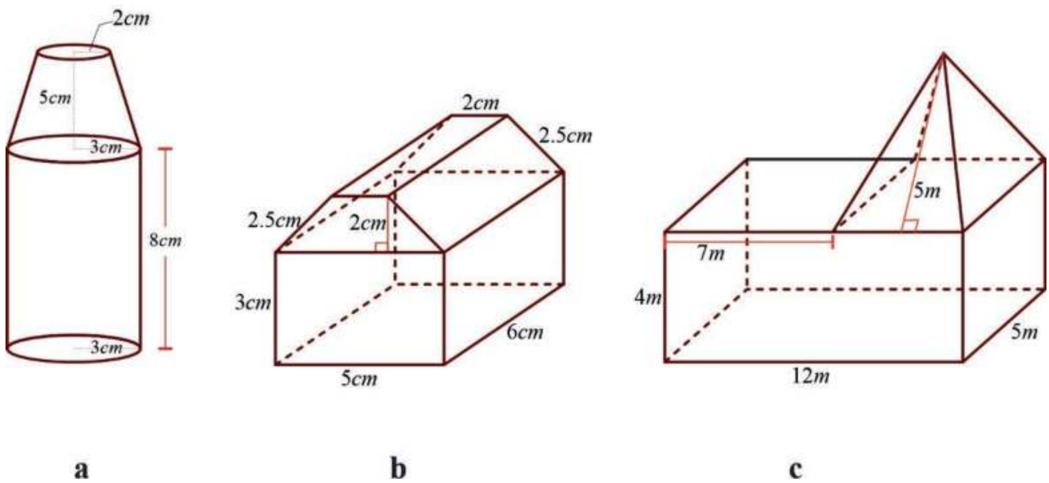



Figure 6.53

2. Hawi bought a new pencil like the one shown in Figure 6.54 on the right. She used the pencil every day in her mathematics class for a week, and now her pencil looks like the one shown on the right. How much of the pencil, in terms of volume did she use?

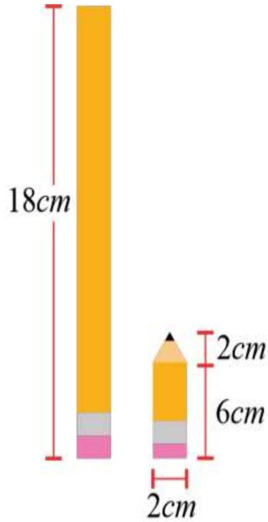


Figure 6.54

## Example 3

A right circular cylinder whose base radius is 10 cm and whose height is 12 cm is drilled a triangular prism hole whose base has edge 3 cm, 4 cm and 5 cm as shown in Figure 6.55. Find the total surface area and volume of the remaining solid.

Use $\pi = 3.14$.

**Solution:**

The base of the drilled triangular prism is a right-angled triangle. Why? The total surface area is the sum of the lateral surface areas of the cylinder and prism, and the base area of the cylinder minus the base area of the prism.

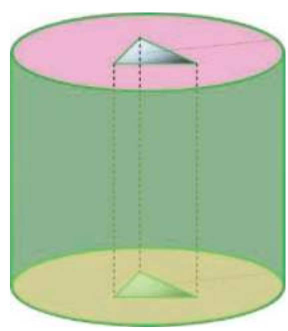


Figure 6.55

TSA $= 2\pi rh + ph + 2\pi r^2 - 2\left(\frac{1}{2}ab\right)$, where $r$ is the radius and $h$ the height of the cylinder, $p$ perimeter of the triangle, $a$ and $b$ are legs of the triangle.

TSA $= 2\pi(10)(12) + (3+4+5)(12) + 2\pi(10)^2 - 2\left(\frac{1}{2} \times 3 \times 4\right)$,

$= (440\pi + 132)\ \text{cm}^2 = ((440 \times 3.14) + 132)\ \text{cm}^2$,

$= 1513.6\ cm^2$.

The volume of the resulting solid = (volume of the cylinder) - (volume of the prism)

$$= \pi r^2 h - \frac{1}{2}abh$$

$$= 1200\pi - 72 = (1200 \times 3.14) - 72 = 3696\ cm^3.$$

## Exercise 6.16

1. A cone is contained in a cylinder as shown in Figure 6.56 so that their base diameter and height have the same length $x$ cm. Calculate the volume of the space inside the cylinder but outside the cone.

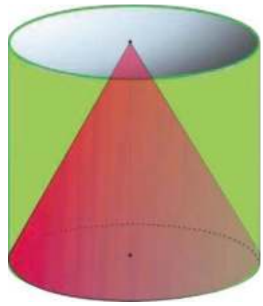


Figure 6.56

2. From a hemispherical solid of radius 8 cm, a conical part is removed as shown in Figure 6.57a. Find the volume and the total surface area of the resulting figure.
3. The altitude of a frustum of a right circular cone is 20 cm and the radius of its base is 6 cm. A cylindrical hole of diameter 4 cm is drilled through the cone with the center of the drill following the axis of the cone, leaving a solid as shown in Figure 6.57b. Find the volume and the total surface area of the resulting solid.
4. Figure 6.57c shows a hemispherical shell. Find the volume and total surface area of the solid.

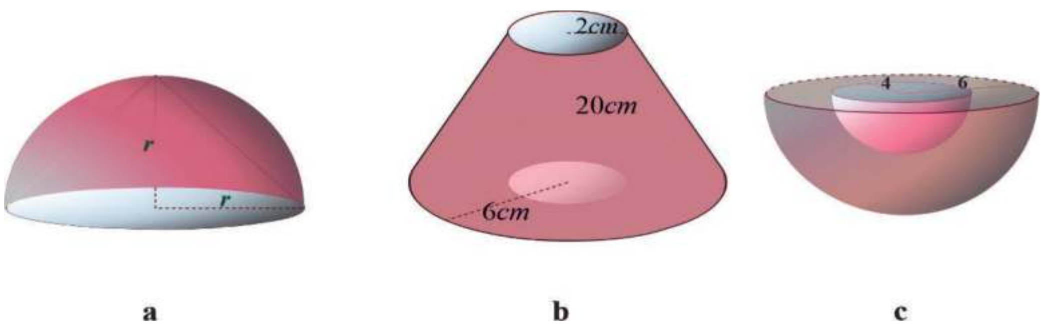



Figure 6.57

5. A cylindrical piece of wood of radius 8 cm and height 18 cm has a cone of the same radius. scooped out of it to a depth of 9 cm. Find the ratio of the volume of the wood scooped out to the volume of the wood which is left. (See Figure 6.58)

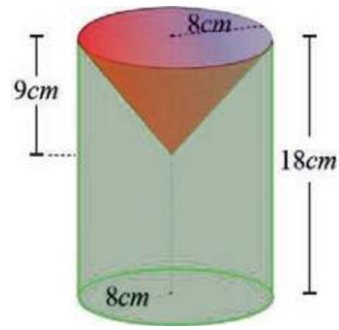



Figure 6.58

## 6.5 Applications

In daily life, volume and surface area could help

- With travel, knowing how much your container can hold could help you use your space most efficiently. For example, the volume of the trunk of a car, a bag or a box.
- Without volume, you can't figure out density, or capacitance and many other things in science.
- Suppose you're manufacturing an object whose shape is a cone, for example, a funnel. You need to know the surface area to determine how much material goes into each cone which determines your material cost.
- When you want to rent an apartment/house, you need to know how much surface area you are getting for your money. This can be broken down to usable space (kitchen, bedrooms, bathrooms), and even storage or extra space (like basement and balconies).
- You could use surface area to find out how much cardboard was used to make a box, or how much fabric was used to make a pillow.

### Example 1

The bottom and top diameters of a backet are 10 cm and 13 cm respectively. If the slant height is 20 cm,

**a.** How many liters of water can it hold?

**b.** How many square centimeters of material are used to make this bucket ignoring the thickness of the backet?

**c.** If 100 $\text{cm}^2$ of the material costs 50 birr then find the total cost of the material. Use $\pi = 3.14$ and put your answer by rounding it to two decimal places.

**Solution:**

**a.** The bucket has the shape of frustum of a cone with base radius $r' = 10$ cm , $r = 13$ cm and slant height $l = 20$ cm. To find the height $h$ of the bucket consider Figure 6.59.

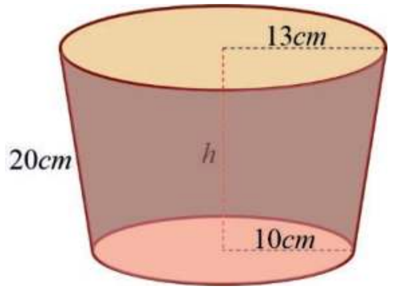



Figure 6.59

$$h^2 + 3^2 = 20^2$$

$$h = \sqrt{20^2 - 3^2} = \sqrt{391} \text{ cm}$$

$$v = \frac{1}{3}\pi h(r^2 + (r')^2 + rr')$$

$$= \frac{1}{3}\pi\sqrt{391}(13^2 + (10)^2 + (13)(10))$$

$$= \frac{1}{3}\pi \times 399\sqrt{391} \text{ cm}^3$$

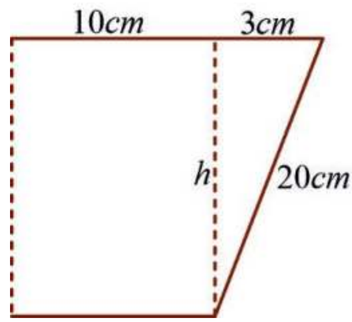



Figure 6.60

Since 1 liter is equal to 1000 $\text{cm}^3$, the bucket can hold $\frac{\frac{1}{3}\times 399\pi\sqrt{391}}{1000} = 8.26209$ liters

8.26209 rounded to 2 decimal places would be 8.26.

Therefore, the answer is 8.26 liters.

**b.** $\text{LSA} = l\pi(r + r') = 20\pi(13 + 10) = 460\pi \text{ cm}^2$

Since the bucket is closed from bottom $\text{BA} = \pi(r')^2 = 100\pi \text{ cm}^2$

$\text{TSA} = \text{LSA} + \text{BA} = 560\pi \text{ cm}^2 = 560 \times 3.14 = 1758.4 \text{ cm}^2$

**c.** If 100 $\text{cm}^2$ of the material costs 50 birr, then 1758.4 $\text{cm}^2$ of material cost

$$\frac{1758.4 \times 50}{100} = 879.2 \text{ birr}$$

### Example 2

A construction company wants to make a water tank in the form of a right circular cylinder that can hold 100,000 liters of water and has height of length 8 m.

**a.** Find the radius of the cylinder that will be constructed.

**b.** If the company wants to paint the lateral surface of the cylinder with anti-rust paint with thickness 2 cm, find the cost of the anti-rust paint at the rate of 15 birr per liter ignoring the thickness of the material used to make the water tank.

**c.** If the company has to pay for a painter only for the lateral surface, find the amount in birr that the company has to pay for the painter at the rate of 50 birr per $m^2$.

Use $\pi = 3.14$ and put your answer by rounding it to two decimal places.

**Solution:**

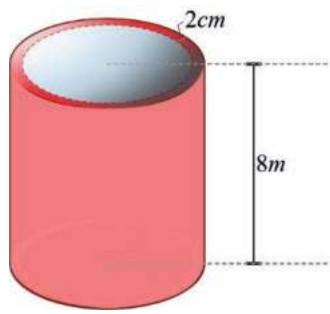



Figure 6.61

**a.** Volume of the water tank = 100,000 liters $= 100\ m^3$

Volume of a cylinder = $\pi r^2 h = 8\pi r^2$

$100 = 8\pi r^2$

$r = \sqrt{\frac{100}{8\pi}}\ m \approx 2\ m$

Therefore, the base radius of the water tank is 2 m.

**b.** $2\ cm = 0.02\ m$, therefore, the radius of the outer circle is 2.02 m and the radius of the inner circle is 2 m.

The volume of the paint = $\pi h(r^2 - (r')^2) = 8\pi(2.02^2 - 2^2)$

$$= 2.019648\ m^3 = 2{,}019.648 \text{ liters}$$

Since the cost of the paint per liter is 15 birr, the total cost of the anti-rust paint is $15 \times 2{,}019.648 = 30{,}294.72$ birr.

**c.** $LSA = 2\pi rh = 2\pi \times 2.02 \times 8 = 101.48\ m^2$

The company has to pay $101.48 \times 50 = 5{,}074.24$ birr for the painter.

## Exercise 6.17

1. A concrete beam is to rest on two concrete pillars. The beam is a cuboid with sides of length 0.6 m, 4 m and 0.5 m. The pillars have diameter 0.5 m and height 2.5 m. Calculate the total volume of concrete needed to make the beam and the pillars. Use $\pi = 3.14$ and put your answer by rounding it to two decimal places.

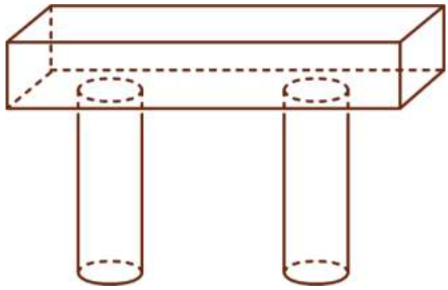


Figure 6.62

2. The diagram shows the cross-section of a pipe of length 50 m. The inner diameter of the pipe is 20 cm and the outer diameter is 28 cm.
   a. Calculate the volume of metal needed to make the pipe. Use $\pi = 3.14$
   b. Calculate the total surface area of the pipe, including the inside surface. Use $\pi = 3.14$

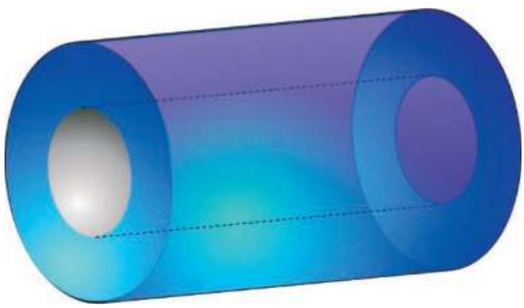


Figure 6.63

3. A lead bar of length 15 cm, width 8 cm and thickness 5 cm is melted down and made in five equal spherical ornaments. Find the radius of each ornament. (Use $\pi = 3.14$)

# Summary

## Prism

$\text{LSA} = ph$

$\text{TSA} = 2\text{BA} + \text{LSA}$

$V = (\text{BA})h$

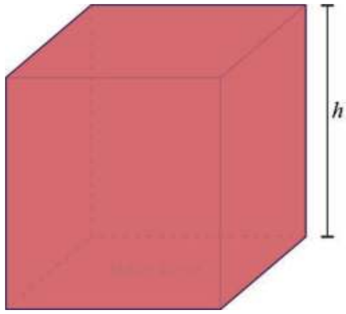



Figure 6.64

## Right circular cylinder

$\text{LSA} = 2\pi rh$

$\text{TSA} = 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

$\text{V} = \pi r^2 h$

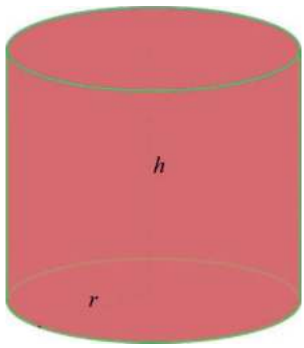



Figure 6.65

## Regular Pyramid

$\text{LSA} = \frac{1}{2}pl$

$\text{TSA} = \text{BA} + \frac{1}{2}pl$

$\text{V} = \frac{1}{3}(\text{BA})h$

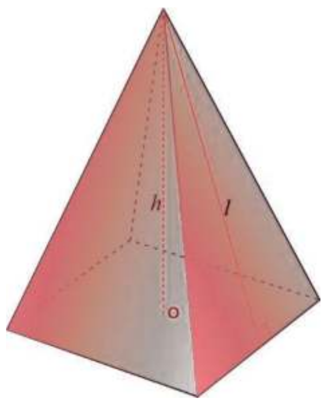



Figure 6.66

## Right circular cone

$\text{LSA} = \pi r l$

$\text{TSA} = \pi r^2 + \pi r l = \pi r(r + l)$

$\text{V} = \frac{1}{3}\pi r^2 h$

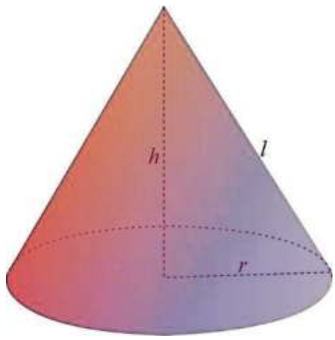



Figure 6.67

## Sphere

$\text{A} = 4\pi r^2$

$\text{V} = \frac{4}{3}\pi r^3$

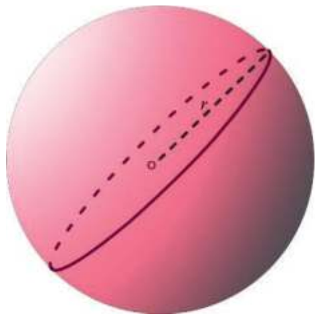



Figure 6.68

## Frustum of a pyramid

$\text{LSA} = \frac{1}{2} l(p + p')$

$\text{TSA} = \frac{1}{2} l(p + p') + A' + A$

$\text{V} = \frac{1}{3} h(\text{A} + \text{A}' + \sqrt{\text{AA}'})$

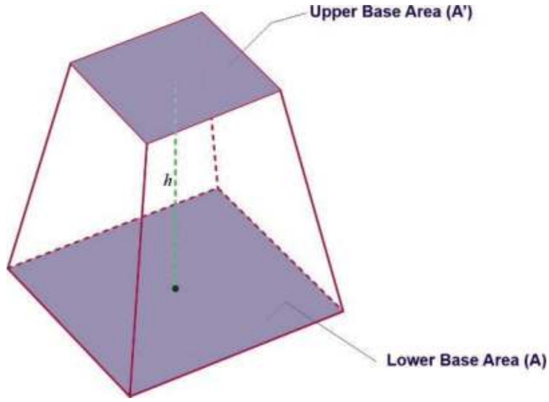



Figure 6.69

## Frustum of a cone

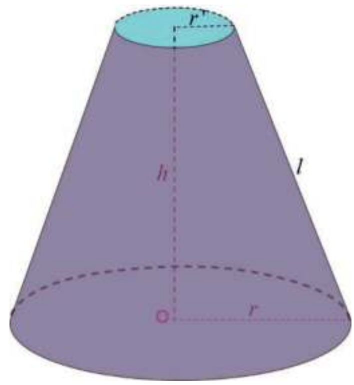



Figure 6.70

$$\text{LSA} = \frac{1}{2}l(2\pi r + 2\pi r') = l\pi(r + r')$$

$$\text{TSA} = \frac{1}{2}l(2\pi r + 2\pi r') + \pi r^2 + \pi r'^2$$

$$= l\pi(r + r') + \pi(r^2 + r'^2)$$

$$\text{V} = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$$

# Review Exercise

**1.** Find the lateral surface area and volume of the following figures.

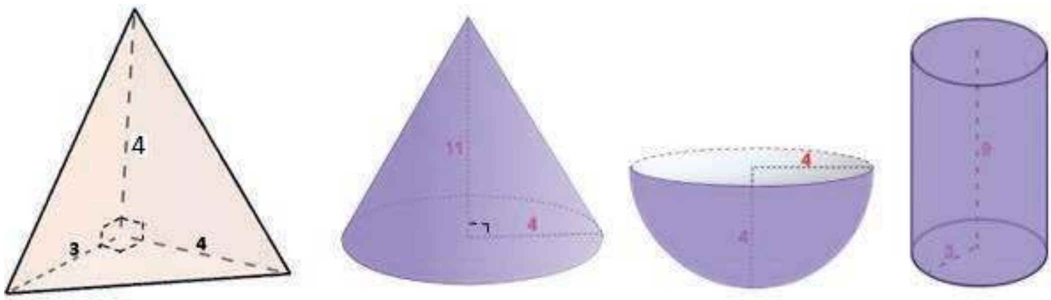



Figure 6.71

**2.** A lateral edge of a right prism is 6 cm and the perimeter of its base is 36 cm. Find the area of its lateral surface.

**3.** The height of a circular cylinder is equal to the radius of its base. Find its total surface area and its volume. Give your answer in terms of its radius.

**4.** Find the total surface area of a regular hexagonal pyramid, given that an edge of the base is 8 cm and the altitude is 12 cm.

**5.** When a lamp of stone is submerged in a rectangular water tank whose base is 20 cm by 50 cm, the water level rises by 1cm. What is the volume of the stone?

**6.** The altitude and base radius of a right circular cone are 5 cm and 8 cm respectively. Find the total surface area and volume of the cone.

**7.** What is the lateral surface area of a regular square pyramid whose base is 12 cm on a side and whose slant height is 10 cm? If a plane is passed parallel to the base and 4 cm from the vertex, what is the lateral area of the frustum?

**8.** The radii of the internal and external surfaces of a hollow spherical shell are 3 m and 5 m respectively. If the same amount of material were formed into a cube what would be the length of the edge of the cube?

**9.** Find the ratio relation between the volumes and the lateral surface areas of the cylinder, sphere and cone, when their heights and diameters are equal.

**10.** A solid is made up of a right circular cone, right circular cylinder and hemisphere, as shown in Figure 6.72. Find the surface area of the composed solid.

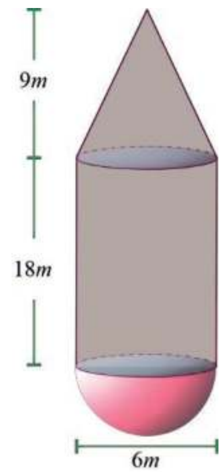



Figure 6.72

# UNIT 7

# COORDINATE GEOMETRY

## Unit Outcomes

By the end of this unit, you will be able to:

- Find the distance between any two given points in the coordinate plane.
- Divide a given line segment into different ratios.
- Describe the equation of a line in different forms.
- Describe the equation of a circle in different forms.
- Relate the slope of parallel and perpendicular lines.

## Unit Contents


- ✓ Point-slope form
- ✓ Slope (gradient)
- ✓ Inclination of a line
- ✓ Two-point form
- ✓ Slope-intercept form
- ✓ Mid-point
- ✓ Equation of a line
- ✓ General Equation of a line
- ✓ Steepness
- ✓ Horizontal line
- ✓ Non-vertical line
- ✓ Coordinates
- ✓ Angle of inclination
- ✓ Coordinate geometry

# INTRODUCTION

Coordinate geometry is one of the most important and exciting ideas of mathematics. In particular, it is central to mathematics students. It provides a connection between algebra and geometry through graphs of lines and curves. This enables geometric problems to be solved algebraically and provides geometric insights into algebra.

The number plane (Cartesian plane) is divided into four quadrants by two perpendicular axes called the $x$-axis (horizontal line) and the $y$-axis (vertical line). These axes intersect at a point called the origin. The position of any point in the plane can be represented by an ordered pair of numbers $(x, y)$. These ordered pairs are called the coordinates of the point. The origin is denoted by the ordered pair (0, 0).

The point with coordinates $(2, 3)$ has been plotted on the Cartesian coordinate plane as shown in figure 7.1. Once the coordinates of two points are known, the distance between the two points and midpoint of the interval joining them can be found.

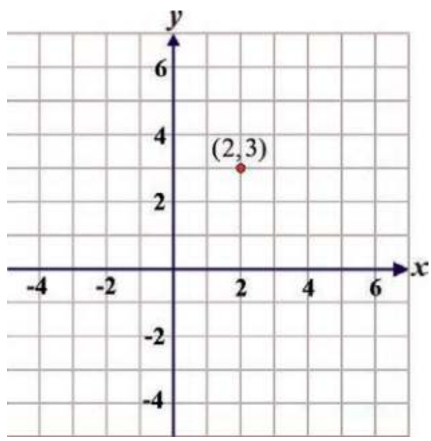



Figure 7.1

## 7.1 Distance Between Two Points

In unit one, you have discussed the Cartesian coordinate plane and you have seen that there is one-to-one correspondence between the set of points in the plane and the set of all ordered pairs of real numbers.

Distance is always positive, or zero if the points coincide. The distance from point $A$ to $B$ is the same as the distance from point $B$ to $A$ (see figure7.2). We first find the distance between two points that are either vertically or horizontally aligned.

The following activity will help you to review the facts you discussed in Grade 9.

### Activity 7.1

1. Consider the number line given in figure 7.2.

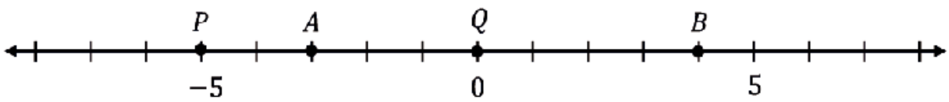



Figure 7.2

   i. Find the corresponding value of points $P, Q, A$ and $B$.

   ii. Find the distance between points a) $A$ and $B$ b) $Q$ and $B$

2. On a number line, the two points $P$ and $Q$ have coordinates $x_1$ and $x_2$, then

   a. find the distance between $P$ and $Q$.

   b. find the distance between $Q$ and $P$.

   c. discuss the relationship between your answers in $a$ and $b$ above.

   d. find $|x_1 - x_2|$ and $|x_2 - x_1|$. What do you observe?

3. How do you plot the coordinates of points in the coordinate plane?

4. Let $S(4, 5)$ and $T(4, 9)$ be points on the coordinate plane.

   a. Plot the points $S$ and $T$.

   b. Is the line through points $T$ and $S$ vertical or horizontal? Why?

## Example 1

**1.** Find the distance between the following pairs of points.

**a.** $A(1, 2)$ and $B(4, 2)$ **b.** $P(1, -2)$ and $Q(1, 3)$

## Solution:

**a.** Since $AB$ is horizontal line, distance $d = |x_2 - x_1| = |4 - 1| = 3$.

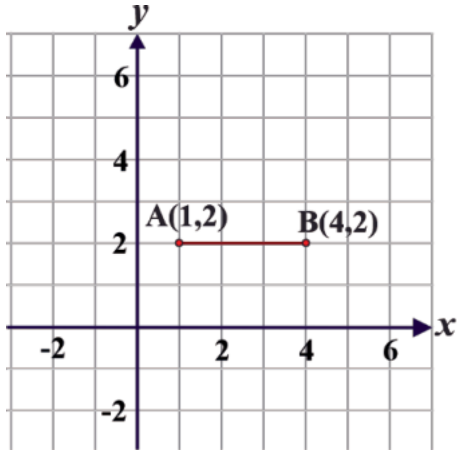



Figure 7.3

**b.** Since $PQ$ is vertical line,

Distance $d = |y_2 - y_1| = |3 - (-2)| = 5$.

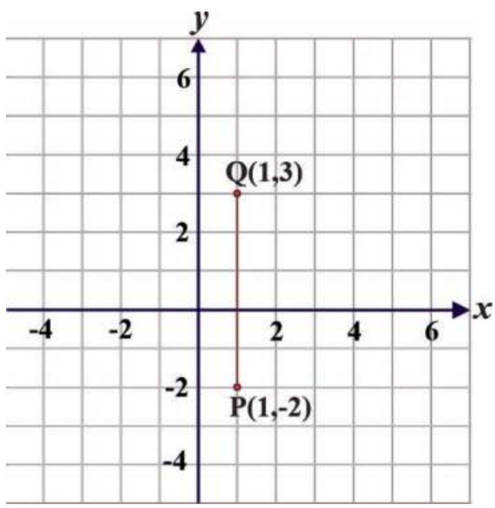



Figure 7.4

## Exercise 7.1

Find the distance between the following points.

a. $A(2,3), B(5,3)$ b. $M(-3,-2), N(2,-2)$ c. $P(4,1), Q(4,-2)$

The example above considers the special cases when the line interval $AB$ is either horizontal or vertical. Pythagoras Theorem is used to calculate the distance between two points when the line interval between them is neither vertical nor horizontal. The distance between the points $A(1,2)$ and $B(4,6)$ is calculated below.

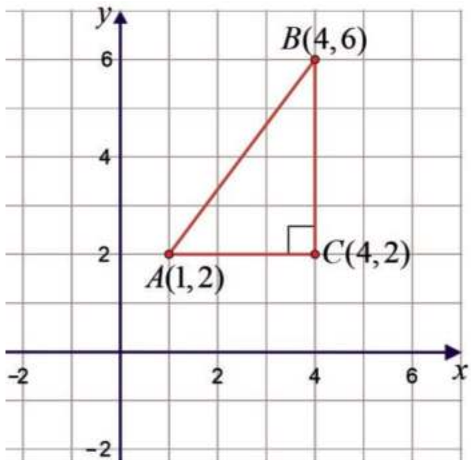



Figure 7.5

$AC = 4 - 1 = 3$ and $BC = 6 - 2 = 4$.

By Pythagoras Theorem,

$$(AB)^2 = (AC)^2 + (BC)^2$$

$$(AB)^2 = 3^2 + 4^2$$

$$= 25$$

Since $AB > 0$, $AB = \sqrt{25} = 5$

So, the distance between $A$ and $B$, $AB = 5$units.

Now, we can obtain a general formula for the length of any interval.

Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points on the $xy$-plane as shown in Figure 7.6.

To find the distance between the points $P$ and $Q$, draw a line passing through $P$ parallel to the $x$-axis and draw a line passing through $Q$ parallel to the $y$-axis. The vertical line and the horizontal line intersect at $R(x_2, y_1)$.

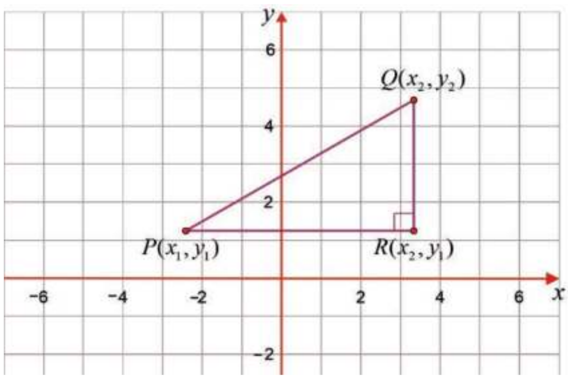



Figure 7.6

The distance PQ can be defined as follows.

Note that ΔPQR is a right-angled triangle and by the Pythagoras Theorem, we have

$$(PQ)^2 = (PR)^2 + (RQ)^2 \text{ . So, } PQ = \sqrt{(PR)^2 + (RQ)^2}.$$

Since distance of PR = ằ$x_2 - x_1$àand distance of RQ = ằ$y_2 - y_1$à

## Definition 7.1

If $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ are endpoints of a line segment $PQ$, then the distance of $\overline{PQ}$, denoted by $d$, is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

### Example 1

Find the distance between the given points.

**a.** $A(2,1)$ and $B(8,9)$ **b.** $P(9,13)$ and $Q(4,1)$

**c.** $R(0,\ -1)$ and $S(-3,3)$

**Solution:**

**a.** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(8-2)^2 + (9-1)^2}$

$= \sqrt{6^2 + 8^2}$

$= \sqrt{100} = 10$

Therefore, the distance between points $A$ and $B$ is 10 units.

**b.** Similarly, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4-9)^2 + (1-13)^2}$

$= \sqrt{(-5)^2 + (-12)^2}$

$= \sqrt{169} = 13$

Therefore, the distance between point $P$ and $Q$ is 13 units.

**c.** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-3-0)^2 + (3-(-1))^2}$

$= \sqrt{(-3)^2 + (4)^2}$

$= \sqrt{25} = 5$

Therefore, the distance between point $R$ and $S$ is 5 units.

## Exercise 7.2

In each of the following, find the distance between the two given points.

a. $A(2,5)$ and $B(4,7)$

b. $P(-3,5)$ and $Q(4,10)$

c. $R(9,5)$ and $S(6, -3)$

d. $M(-4,-3)$ and $N(5,7)$

e. $T(-5,-2)$ and $S(0,-14)$

f. The origin and a point $P(\sqrt{2}, \sqrt{2})$.

# 7.2 Division of a Line Segment

### Activity 7.2

a. What is a line?

b. What is the difference between a line and a line segment?

c. Can you divide 10 cm thread into two equal parts?

d. Discuss the midpoint of a line segment.

e. Define the ratio of two numbers or line segments.

A line segment can be divided into '$n$' equal parts, where '$n$' is any natural number.

**For example**; a line segment of length 10 cm is divided into two equal parts by using a ruler as follows:

Mark a point 5 cm away from one end, 10 cm is divided into two 5 cm line segments. Similarly, a line segment of length 15 cm can be divided in the ratio 2:1. Let $AB$ is the line segment of length 15 cm and $C$ divides the line in the ratio 2:1 as shown in figure 7.7.

If $CB = x$, then $AC = 2x$. So, $AC + CB = 2x + x = 15$ implies $x = 5$ cm. Then, $AC = 10$ cm and $CB = 5$ cm


Figure 7.7

Given a line segment PQ, let us find the coordinates of R, dividing the line segment PQ internally in the ratio p: q, i.e. $\frac{PR}{RQ} = \frac{p}{q}$, where p and q are given positive numbers.

Let the coordinate of $R$ be $(x_0, y_0)$ and the coordinate of $P(x_1, y_1)$ and $Q(x_2, y_2)$ with $x_1 \neq x_2$ and $y_1 \neq y_2$.

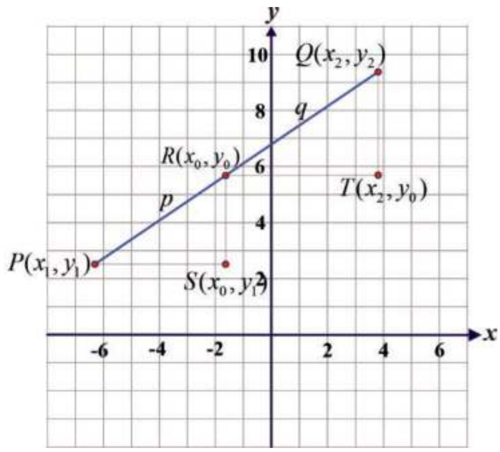



Figure 7.8

As shown in figure 7.8, $PS = |x_0 - x_1|$, $RT = |x_2 - x_0|$, $SR = |y_0 - y_1|$ and $TQ = |y_2 - y_0|$

Since $\Delta PSR \approx \Delta RTQ$.

$$\frac{PS}{RT} = \frac{PR}{RQ} \text{ and } \frac{SR}{TQ} = \frac{PR}{RQ}. \text{ So, } \frac{x_0 - x_1}{x_2 - x_0} = \frac{p}{q} \text{ and } \frac{y_0 - y_1}{y_2 - y_0} = \frac{p}{q} \text{ (since } \frac{PR}{RQ} = \frac{p}{q}).$$

Now, solve for $x_0$ and $y_0$

$$\frac{x_0 - x_1}{x_2 - x_0} = \frac{p}{q}, \text{ i.e., } q(x_0 - x_1) = p(x_2 - x_0) \text{ and } \frac{y_0 - y_1}{y_2 - y_0} = \frac{p}{q},$$

i.e., $q(y_0 - y_1) = p(y_2 - y_0)$

$$qx_0 - qx_1 = px_2 - px_0 \text{ and } qy_0 - qy_1 = py_2 - py_0$$

$$x_0 = \frac{px_2+qx_1}{p+q} \text{ and } y_0 = \frac{py_2+qy_1}{p+q}.$$

## The Section Formula

The point $R(x_0, y_0)$ dividing the line segment PQ internally in the ratio $p:q$ is given by:

$$R(x_0, y_0) = (\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q}),$$

where $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the end-points.

This is called the section formula.

### Example 1

A point divides internally the line segment joining the points $A(-7, 4)$ and $B(8, 9)$ in the ratio $3:2$. Find the coordinates of the point.

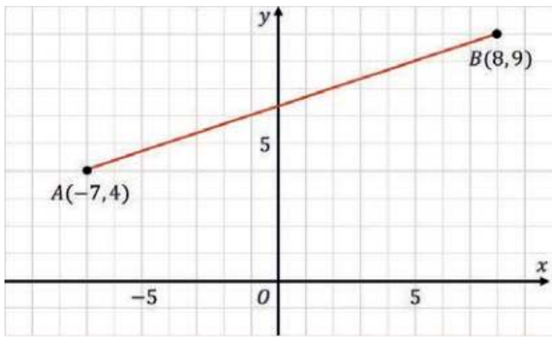



Figure 7.9

**Solution:**

Given: $p = 3$ and $q = 2$.

Let $R(x_0, y_0)$ be the point where $(x_0, y_0)$ are coordinates of the point which divides internally the line-segment joining the given points in the given ratio.

$$\text{Then, } R(x_0, y_0) = (\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q})$$

$$= (\frac{3 \times 8 + 2 \times (-7)}{3+2}, \frac{3 \times 9 + 2 \times 4}{3+2}) = (2, 7)$$

Therefore, the coordinates of the required point $R(x_0, y_0) = (2, 7)$.

## Exercise 7.3

1. Find the point dividing the line segment $AB$ internally in the given ratio.
   a. $A(1,2),\ B(4,5)$, 1:2 .
   b. $A(2,-3),\ B(-1,5)$, 3:1.
2. A line segment has end points $P(-1,5)$ and Q (5, 2). Find the coordinates of the points that trisect the segment.

## The Midpoint Formula

Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points on the $xy$-plane and point $R(x_0, y_0)$ is midpoint of $\overline{PQ}$. Then,

$$R(x_0, y_0) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

### Example 1

Find the midpoint $M$ of the line segment joining the origin and the point (4, 0).

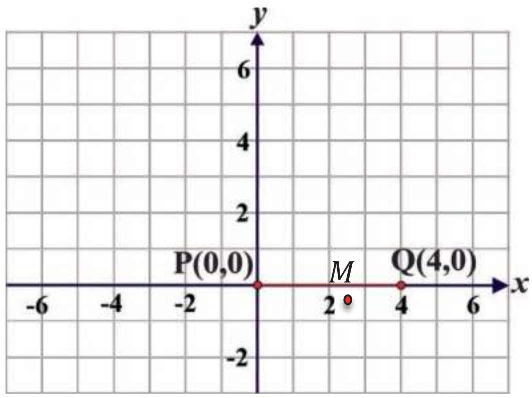



Figure 7.10

**Solution:**

It is easy to see that this line is 4 units in length and its midpoint is $(2,0)$. This makes it easy to illustrate how the midpoint formula works.

First, let's represent the origin, $P(0,0)$ as $(x_1, y_1)$ and the point $Q(4,0)$ as $(x_2, y_2)$. Then if $R(x, y)$ is the midpoint, then we can substitute them into the midpoint formula as to find this point as

$$R(x_0, y_0) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{0+4}{2}, \frac{0+0}{2}\right) = (2,0).$$

## Example 2

Find the midpoint of the line segment joining the points $A(-1,-2)$ and $B(3,2)$.

**Solution:**

Midpoint of $\overline{AB}$

$(x_{0}, y_0) = \left(\frac{-1+3}{2}, \frac{-2+2}{2}\right) = (1,0).$

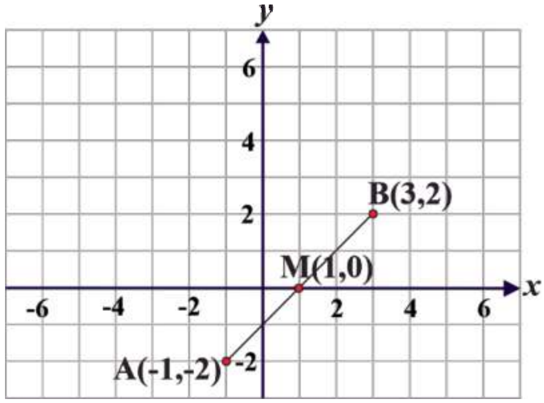



Figure 7.11

## Exercise 7.4

1. Find the coordinate of the midpoint of the line segments joining the points:
   a. $P(1,-3)$ and Q (4, 5) b. $P(-9,-3)$ and Q (18, 2)
2. Find the midpoint of the sides of the triangle with vertices $A(-1,3), B(4,6)$ and $C(3,-1)$.
3. If $M(4,6)$ is the midpoint of the line segment $AB$, point $A$ has the coordinates $(-3,-2)$. Find the coordinates of point $B$.
4. Find the coordinates of point $C\ (x, y)$ where it divides the line segment joining $(4,-1)$ and $(4,3)$ in the ratio 3: 1 internally.

# 7.3 Equation of a Line

In this subtopic, we find the equation of a straight line, when we are given some information about the line. The information could be the value of its gradient, together with the coordinates of a point on the line. Alternatively, the information might be the coordinates of two different points on the line. There are several different ways of expressing the final equation, and some are more general than others.

## 7.3.1 Gradient (slope) of a line

From your everyday experience, you might be familiar with the idea of gradient (slope). A hill may be steep or may rise very slowly. The number that describes the steepness of a hill is called the gradient (slope) of the hill. We measure the gradient of the hill by the ratio of the vertical rise to the horizontal run as shown in figure 7.12.


Figure 7.12

**Activity 7.3**

Given points $A(-4, 2)$, $B(7, 5)$ and $C(-3, 8)$ as shown in the Figure 7.13,

a. Find the value of $\frac{y_2 - y_1}{x_2 - x_1}$ when,

(i) $A$ and $B$, (ii) $A$ and $C$.

b. Are the values obtained in (i) and (ii) above equal?

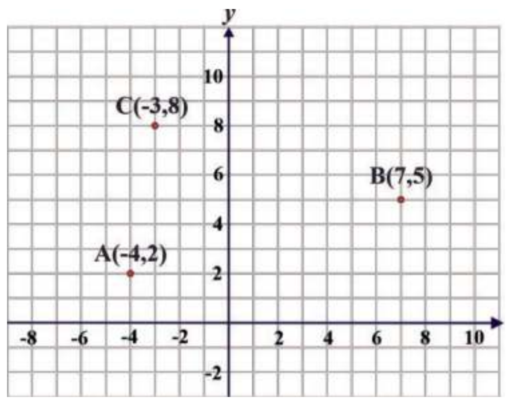



Figure 7.13

In coordinate geometry, the gradient of a non-vertical straight line is the ratio of "change in $y$-coordinates" to the corresponding "change in $x$-coordinates".

That is, the slope of a line through points $P$ and $Q$ is the ratio of the vertical distance from $R$ to $Q$ to the horizontal distance from $P$ to $R$. See Figure 7.14.

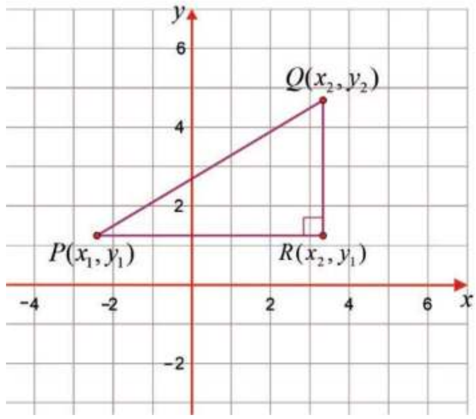



Figure 7.14

If we denote the gradient of a line by the letter $m$, then

$$m = \frac{change\ in\ y\ coordinate}{change\ in\ x\ coordinate} = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2.$$

## Definition 7.2

If $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ are points on a line with $x_1 \neq x_2$, then the gradient of the line, denoted by $m$, is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

## Example 1

**1.** Find the gradient of the line passing through each of the following pairs of points:

**a.** $P(2, 1)$ and Q (4, 7)

**b.** $P(5, -1)$ and Q (6, 9)

**c.** $P(8, 3)$ and Q (3, 13)

**d.** $P(7, 3)$ and Q (-4, 3)

**e.** $P(2, 3)$ and Q (2, -7)

**Solution:**

**a.** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-1}{4-2} = \frac{6}{2} = 3$

**b.** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-(-1)}{6-5} = 10$

**c.** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13-3}{3-8} = -2$

**d.** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{-4-7} = \frac{0}{-11} = 0$ (Since the line is horizontal (parallel to $x$-axis) it has zero slope.)

**e.** Since the line is a vertical line (parallel to $y$-axis) it has no slope.

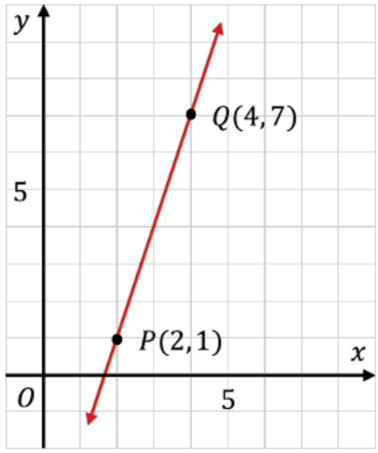



Figure 7.15

## Exercise 7.5

1. Find the gradient of the line passing through the following points.
   a. $P(4, 3)$ and $Q(6, 7)$
   b. $P(4, -3)$ and $Q(7, -4)$
   c. $P(0, 3)$ and $Q(0, -7)$
   d. $P(-6, -3)$ and $Q(-2, 5)$
2. If $A(-4, 6)$, $B(-1, 12)$ and $C(-7, 0)$ are points, then show that they are collinear.
3. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are distinct points on a line with $x_1 = x_2$, then what can be said about the gradient of the line? Is the line vertical or horizontal?
4. Consider the line with equation $y = x + 4$. Take three distinct points $A$, $B$ and $C$ on the line $y = x + 4$.
   a. Find the gradient using $A$ and $B$.
   b. Find the gradient using $A$ and $C$.
   c. What do you observe from $a$ and $b$?

## 7.3.2 Slope of a line in terms of angle of inclination

The angle measured from the positive $x$-axis to a line, in the anticlockwise direction is called the inclination of the line or the angle of inclination of the line. This angle is always less than $180^\circ$.

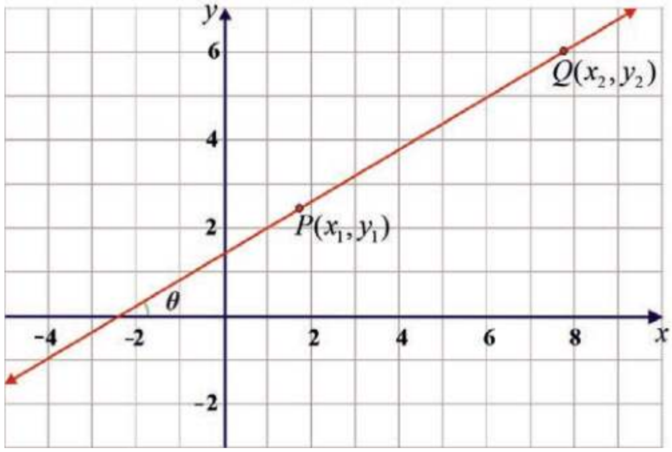



Figure 7.15

## Activity 7.4

Consider the right-angled $\Delta OPQ$ in Figure 7.16.

a. How long is the hypotenuse?

b. What is the tangent of $\angle POQ$ ?

c. By finding the coordinates of points $P$ and $Q$, calculate the slope of the line.

d. What relationship do you see between your answers for questions $b$ and $c$**?**

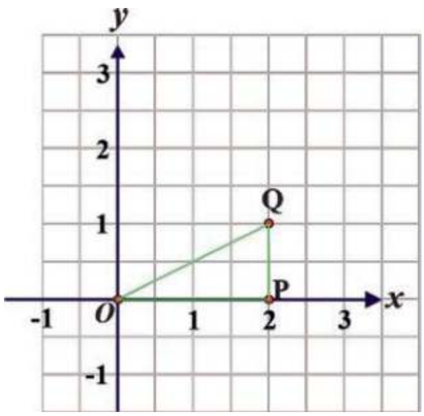



Figure 7.16

The above activity will help you to understand the relationship between slope and angle of inclination. For a non-vertical line, the tangent of this angle is the slope of the line.

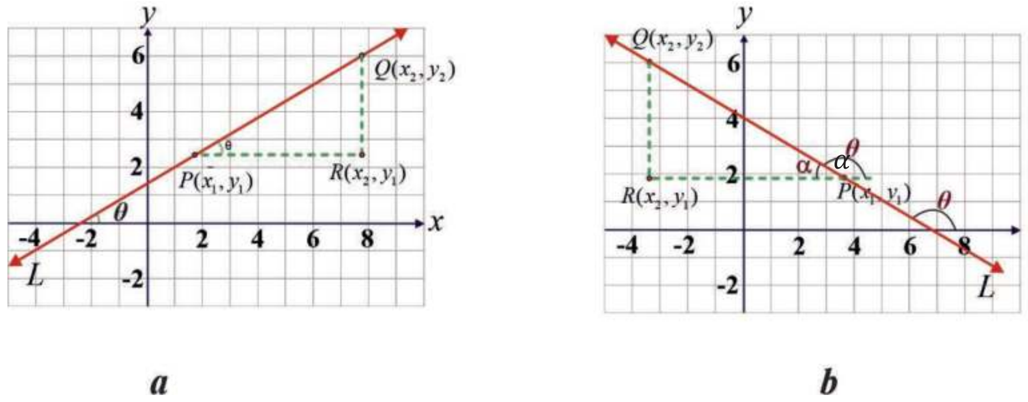



Figure 7.17

In the Figure 7.17 (a), the slope of the straight line $L$ is

$$m = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\angle RPQ). \text{ Therefore, } m = \tan\theta.$$

A line making an acute angle of inclination $\theta$ with the positive direction of the x-axis has positive slope.

Similarly, a line with obtuse angle of inclination, (see Figure 7.17b), has negative slope. The slope of the straight line L is

$$m = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta. \text{ Therefore, } m = \tan(180^0 - \alpha) = -\tan\alpha.$$

In general, the slope of a line may be expressed in terms of the coordinates of two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the line as follows:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta, x_1 \neq x_2$, where $\theta$ is the anticlockwise angle between the positive $x$-axis and the line $L$.

### Example 1

Find the slope of a line if its angle of inclination is:

**a.** 45° **b.** 120°

**Solution:**

**a.** Slope, $m = \tan\theta = \tan 45^\circ = 1$

**b.** Since $\theta = 120^\circ$, the supplemental angle $\alpha = 60^\circ$.

Thus, $m = -\tan\alpha$.

$= -\tan 60^\circ$

$= -\sqrt{3}.$

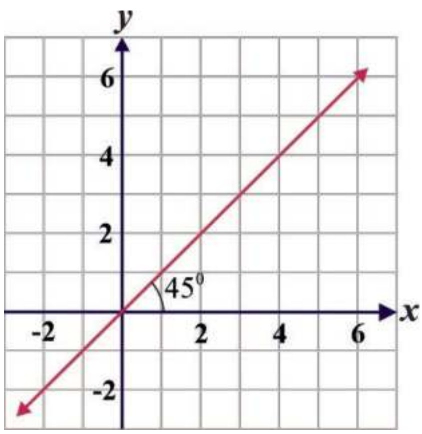



Figure 7.18

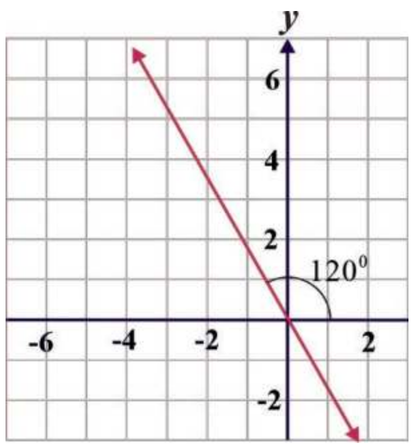



Figure 7.19

## Exercise 7.6

1. Find the slope of a line if its angle of inclination is:

   a. 60°  b. 150°

## 7.3.3 Different forms of equation of a line

The equation for the slope $m$ of a line passing through the point $P(x_1, y_1)$ is called point slope form of equation of a straight line and is given by:

$$y - y_1 = m(x - x_1)$$

The slope-intercept form of equation of the line is given by:

$\boldsymbol{y = mx + b}$, where $b$ is the $y$-intercept. Observe the properties of the line in relation to the slope:

**1)** If $m \in R$ where R is the set of real numbers, then

- $m > 0$, then the line rises from left to right
- $m < 0$, then the line goes downward from left to right
- $m = 0$, then the line is horizontal

**2)** A vertical line has no slope.

**Two points form of equation of a line:**

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the given points on the line $L$ and $P(x, y)$ be any point on the line $L$ as shown in the figure 7.20.

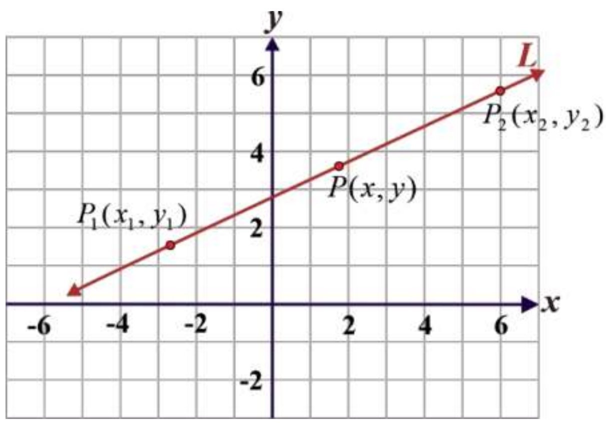



Figure 7.20

From figure 7.20, the three points $P_1$, $P_2$ and $P$ are collinear. Why?

slope of $\overline{PP_1}$ = slope of $\overline{PP_2}$

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}.$$

So, $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$.

Thus, equation of the line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by: $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$.

This is called two-point form of equation of a line.

**Remark:** The general form of the equation of a line is given as $Ax + By + C = 0$, where $A, B,$ and $C$ are real numbers.

### Example 1

Find the equation of a line when $m = 3, b = -1$.

**Solution:**

$y = mx + b$. Thus $y = 3x - 1$

### Example 2

Find the $y$-intercept and the equation of the line with slope $m$, passing through the given point $P$.

**a.** $m = 4; P(-5, 0)$

**b.** $m = -2; P(4, 3)$

**Solution:**

**a.** $m = \frac{y-y_1}{x-x_1}$ and so, $4 = \frac{y-0}{x-(-5)}$, $y = 4(x + 5) = 4x + 20$.

Or $y = mx + b$. $y = 4x + b$.

Since the line passes through $P(-5, 0)$, $b = 0 - 4 \times (-5) = 20$.

Therefore, $y = 4x + 20$.

**b.** $m = \frac{y-y_1}{x-x_1}$ and so, $-2 = \frac{y-3}{x-4}$, $y - 3 = -2(x - 4) = -2x + 11$.

Or $y = mx + b$. $y = -2x + b$.

Since the line passes through $P(4, 3)$, $b = 3 + 2 \times 4 = 11$.

Therefore, $y = -2x + 11$.

### Example 3

If a line passes through the points $(1, 4)$ and $(2, 6)$, then what is the equation of the line? What is the slope and $y$-intercept?

**Solution:**

Let $P(x, y)$ be any point on the line that passes through the points $(1, 4)$ and $(2, 6)$.

We have,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 4 = \frac{6-4}{2-1}(x - 1)$$

$$y - 4 = 2(x - 1)$$

Therefore, the equation of the line is $y = 2x + 2$. The slope is 2 and $y$-intercept is 2.

Or

Let the slope and $y$-intercept be $m$ and $b$. Then, $m = \frac{6-4}{2-1} = 2$.

The equation becomes $y = 2x + b$, then this line passes one of the two points.

So, taking $(1, 4)$, $4 = 2 \cdot 1 + b$, then, $b = 2$

Therefore, the equation of the line is $y = 2x + 2$.

## Exercise 7.7

1. Find the equation of the line with slope $m$ and $y$-intercept b.

   a. $m = -6$; $b = \frac{5}{3}$ 　　 b. $m = 0$; $b = -2$.

   c. $m = -\frac{3}{4}$; $b = \frac{1}{6}$

2. Find the equation of the line with slope $m$ and passing through the given point $P$.

   a. $m = 3$; $P(2, 4)$ 　　 b. $m = -2$; $P(-3, -1)$

   c. $m = \frac{4}{5}$; $P(-5, 0)$ 　　 d. $m = 0$; $P(7, -4)$

3. Find the equation of the line passing through the given points.

   a. $P(1, 3)$ and $Q(3, 7)$ 　　 b. $A(-1, 2)$ and $B(2, -3)$

   c. $R(4, 3)$ and $S(5, -4)$ 　　 d. $P(1, 8)$ and $Q(7, -2)$

   e. $C(6, 3)$ and $D(5, -5)$ 　　 f. $M(-9, 4)$ and $N(-7, -3)$

4. Suppose a line has $x$-intercept $p$ and $y$-intercept $q$ , for $p, q \neq 0$ ;

   Show that the equation of the line is $\frac{x}{p} + \frac{y}{q} = 1$.

5. For each of the following equations, find the slope and $y$-intercept:

a. $5x + 2y + 10 = 0$

b. $\frac{5}{4}x - y = 0$

c. $7x - 4y - 56 = 0$

d. $\frac{1}{3}x + \frac{5}{12}y - \frac{1}{4} = 0$

e. $y - 5 = 0$

6. If a line passing through the points $P(2, 5)$ and $Q(-4, 7)$, then find
   a. the point-slope form of the equation of the line;
   b. the slope-intercept form of the equation of the line;
   c. the two-point form of equation of the line. What is its general form?

## 7.4 Parallel and Perpendicular Lines

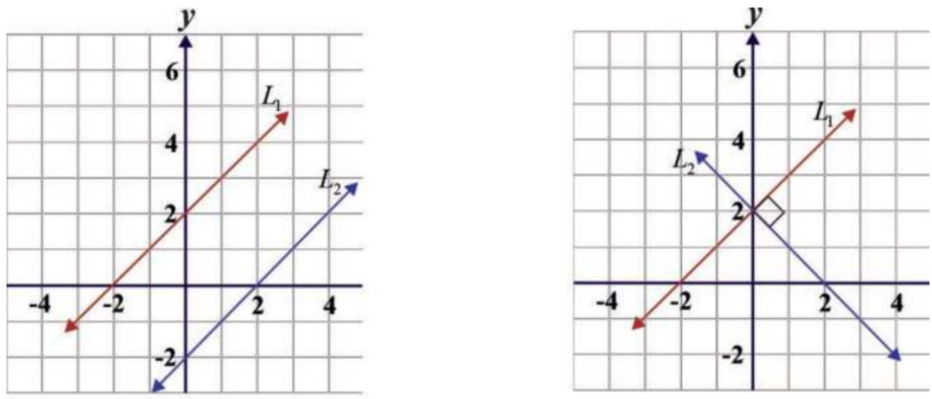
This sub-topic looks at the relationship between the slopes of parallel lines as well as new concept. You will need to know how to find the slope of a line given an equation and how to write the equation of a line. Do the lines intersect or stay apart? If they intersect, do they create a $90^0$? These are the questions we ask about parallel and perpendicular lines.

### 7.4.1 Slopes of parallel and perpendicular lines

This topic looks at the relationship between the slopes of parallel as well as perpendicular lines. Slopes can be used to see whether two non-vertical lines in a plane are parallel, perpendicular, or neither.

**Activity 7.5**

1. Discuss parallel and perpendicular lines.
2. In the figure 7.21 (A), $l_1$ and $l_2$ are parallel.
   a. Calculate the slope of each line.
   b. Find the equation of each line.
   c. Discuss how their slopes are related?




**A** **B**

Figure 7.21

3. In figure 7.21 (B), $l_1$and $l_2$ are perpendicular.
   a. Calculate the slope of each line.
   b. Find the equation of each line.
   c. Discuss how their slopes are related?

## Theorem 7.1

If two non - vertical lines $l_1$ and $l_2$ having slope $m_1$ and $m_2$ respectively are parallel to each other, then they have the same slope ($\mathrm{m}_1 = m_2$).

## Proof:

Suppose you have two non-vertical $l_1$ and $l_2$ with slopes $\mathrm{m}_1$ and $\mathrm{m}_2$ and inclination $\beta$ and $\theta$ , respectively as shown in figure 7.22. If $l_1$and $l_2$ are parallel, then $\theta = \beta$ . Consequently,

$\mathrm{m}_1 = \tan\beta = \tan\theta = \mathrm{m}_2$

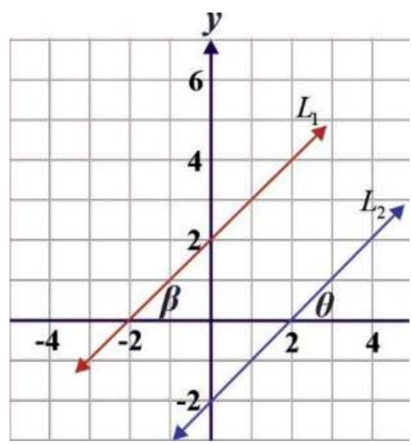



Figure 7.22

### Example 1

Show that the line passing through the points $P(6,4)$ and $R(7,11)$ is parallel to the line passing through $A(0,0)$ and $B(2,14)$.

**Solution:**

Let slopes of line PR and AB be $m_1$ and $m_2$ respectively.

$$m_1 = \frac{11-4}{7-6} = 7,\ m_2 = \frac{14-0}{2-0} = 7.$$

The two lines have the same slope. Therefore, they are parallel.

### Example 2

Find the equation of the line which is parallel to the line $y = -2x + 6$ and passing through the point $P(1,10)$.

**Solution:**

The slope of the line $y = -2x + 6$ is $m = -2$. Therefore, the line through the point $P(1,10)$ parallel to $y = -2x + 6$ has equation $y - y_1 = m(x - x_1)$

$$y - 10 = -2(x - 1)$$

$$y = -2x + 12$$

## Exercise 7.8

1. Show that the line passing through the points $A(7,\ 5)$ and $B(6,11)$ is parallel to the line passing through $P(5,1)$ and $Q(3,13)$.
2. Find the equation of a line parallel to :
   a. $y = 3x - 4$ and passing through point $(2,8)$
   b. $3x + 4y = 5$ and passing through points$(1,1)$

### Theorem 7.2

Two non-vertical lines having slopes $m_1$ and $m_2$ are perpendicular if and only if $m_1.m_2 = -1$.

## Proof:

Suppose line $l_1$ is perpendicular to line $l_2$. Let $m_1$ and $m_2$ be slope of $l_1$ and $l_2$, respectively. Let $R(x_0, y_0)$ be the point of intersection and choose points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on $l_1$ *and* $l_2$, respectively. Draw $\Delta QSR$ and $\Delta RTP$ as shown in the figure 7.23.

$\Delta QSR \approx \Delta RTP$ Why?

$\frac{RS}{QS} = \frac{PT}{RT}$ Why?

$\frac{y_1 - y_0}{x_1 - x_0} = \frac{x_0 - x_2}{y_2 - y_0} = -\frac{x_2 - x_0}{y_2 - y_0}$ Why?

$\frac{y_1 - y_0}{x_1 - x_0} = -\frac{1}{\frac{y_2 - y_0}{x_2 - x_0}}$ Why?

So, $m_1 = -\frac{1}{m_2}$ or $m_1 . m_2 = -1$.

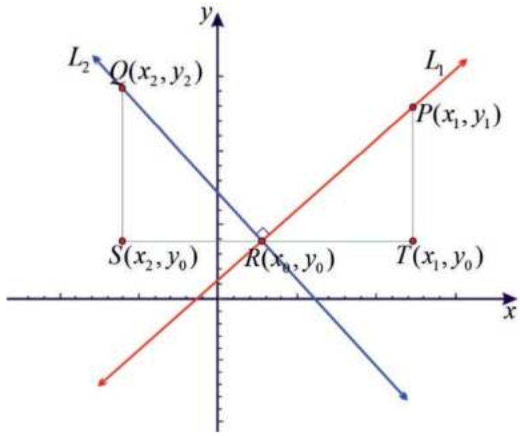



Figure 7.23

Conversely, you could show that if two lines have slopes $m_1$ and $m_2$ with $m_1 . m_2 = -1$ then the lines are perpendicular.

### Example 1

Show that the line passing through the points $A(6, 0)$ and $B(0, 12)$ is perpendicular to the line through $P(8, 10)$ and $Q(4, 8)$.

### Solution:

Let slopes of line AB and PQ be $m_1$ and $m_2$ respectively.

$m_1 = \frac{12-0}{0-6} = -2$ and $m_2 = \frac{8-10}{4-8} = \frac{-2}{-4} = \frac{1}{2}$

Thus, $m_1 \cdot m_2 = -2 \times \frac{1}{2} = -1$.

Therefore, the line that passes through the points $A$ and $B$ is perpendicular to the line that passes through the points $P$ and $Q$.

## Example 2

Find the equation of the line passing through the point $(5, 4)$ and

**a.** parallel to the line $y = 2x + 1$.

**b.** perpendicular to the line $y = 2x + 1$.

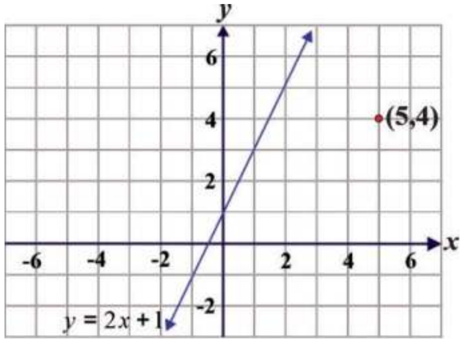



Figure 7.24

### Solution:

**a.** The parallel line needs to have the same slope of $m = 2$. We can solve this using the slope intercept form of equation of a line as

$y - y_1 = m(x - x_1)$

$y - 4 = 2(x - 5)$ (Given the point (5,4))

$y - 4 = 2x - 10$

$y = 2x - 10 + 4$ (Why?)

$y = 2x - 6$ (Why?)

Hence, $y = 2x - 6$ is a line parallel to $y = 2x + 1$.

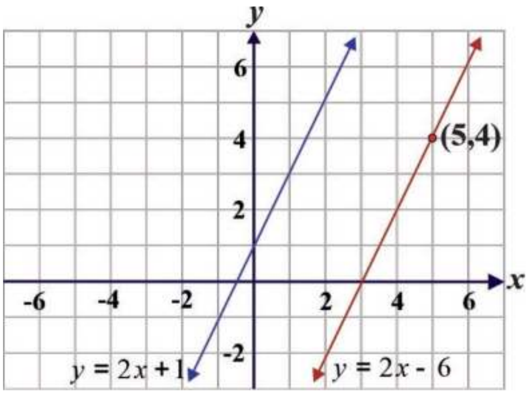



Figure 7.25

**b.** The two lines are perpendicular. So, $m = -\frac{1}{2}$.

By slope intercept form of equation of a line $y - y_1 = m(x - x_1)$

$y - 4 = -\frac{1}{2}(x - 5)$, Since $m = -\frac{1}{2}$ and given the point (5,4)

$$y = -\frac{1}{2}x + \frac{5}{2} + 4 = -\frac{1}{2}x + \frac{13}{2}$$

Hence, the line $y = -\frac{1}{2}x + \frac{13}{2}$ is perpendicular to $y = 2x + 1$, as shown figure 7.26.

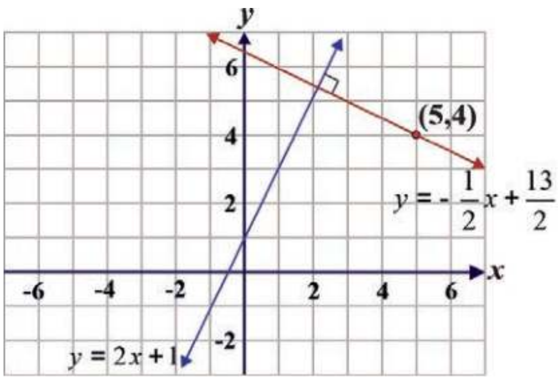



Figure 7.26

## Exercise 7.9

1. Find the slope of a line that is:
   a. parallel to the line : $y = -7x + 5$
   b. perpendicular to the line: $y = -7x + 5$
2. Find the equation of the line that is perpendicular to
   a. $y = -3x + 5$ and passes through the point $(7, 2)$.
   b. $y = 4x + 5$ and passes through the point $(-7, 2)$.

# 7.5 Equation of a Circle

## Activity 7.6

Let's think, what are the set of all points in a plane which are at a distance of 5 units from the origin?

The **equation of a circle** is different from the formulas that are used to calculate the area or the circumference of a circle. This equation is used across many problems of circles in coordinate geometry.

An equation of a circle represents the position of a circle in a Cartesian plane. If we know the coordinates of the center of the circle and the length of its radius, we can write the equation of a circle. The equation of circle represents all the points that lie on the circumference of the circle.

**Definition 7.2**

A circle is the set of all points on a plane with a fixed distance from a fixed point. This fixed point is called the center of the circle and the fixed distance is the radius $r$ of the circle.

## 7.5.1 Different Forms of Equation of Circle

An equation of circle represents the position of a circle on a Cartesian plane. A circle can be drawn on a piece of paper given its center and the length of its radius. Using the equation of circle, once we find the coordinates of the center of the circle and its radius, we will be able to draw the circle on the Cartesian plane. There are different forms to represent the equation of a circle,

- **Standard form**
- **General form**

### Standard Equation of a Circle

A circle is a closed curve that is drawn from the fixed point called the center, in which all the points on the curve are having the same distance from the center point of the center. The equation of a circle with $(h, k)$ center and $r$ radius is given by:

$$(x - h)^2 + (y - k)^2 = r^2$$

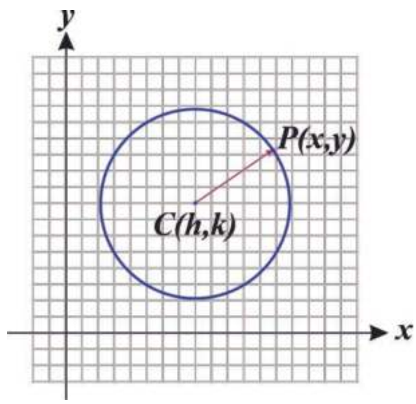



Figure 7.27

## Example 1

Find the equation of the circle whose center and radius are following:

**a.** $C(3,4), r = 2$ **b.** $C(4,-2), r = 5$

**Solution:**

**a.** Given the center $(h,k) = (3,4)$ and $r = 2$, and $(x-h)^2 + (y-k)^2 = r^2$.

$(x-3)^2 + (y-4)^2 = 2^2$

**b.** $(x-4)^2 + (y-(-2))^2 = 5^2$

$(x-4)^2 + (y+2)^2 = 5^2$

### Equation of a Circle When the Centre is Origin

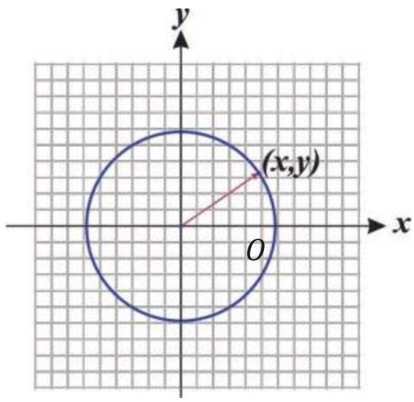



Figure 7.28

Consider an arbitrary point $P(x, y)$ on the circle. Let '$a$' be the radius of the circle which is equal to $OP$.

We know that the distance between the point $(x, y)$ and origin $(0,0)$ can be found using the distance formula which is equal to: $\sqrt{x^2 + y^2} = a$.

Now, squaring on both sides to obtain, $x^2 + y^2 = a^2$.

This is the equation of the circle with the center as the origin.

## Example 2

Consider a circle whose center is at the origin and radius is equal to 6 units.

**Solution:**

Given: Centre is $(0, 0)$, radius is 6 units.

We know that the equation of a circle when the center is origin:

$x^2 + y^2 = a^2$. For the given condition, the equation of the circle is given as

$x^2 + y^2 = 6^2$

$x^2 + y^2 = 36$, which is the equation of the circle.

### Example 3

Show the point (6, 8) is on the circle with equation $x^2 + y^2 = 100$.

**Solution:**

Given: $x = 6$ and $y = 8$. Substitute them into the equation: $x^2 + y^2 = 100$

(LHS) of the equation: $6^2 + 8^2 = 100 \Rightarrow (LHS) = (RHS)$ So, point (6, 8) is on the circle.

## Exercise 7.10

1. Find the equation of the circle in standard form for the following circles.
   a. $C(2, -3), r = 3$.
   b. $C(-7, 4), r = 4$
   c. $C(0, 0), r = \sqrt{10}$
2. Find the center and radius of the following circles.
   a. $(x-2)^2 + (y-5)^2 = 7^2$.
   b. $x^2 + y^2 = 100$.
   c. $(x+1)^2 + y^2 = 3^2$.
3. Show the point (-12, 5) is on the circle with equation $x^2 + y^2 = 169$.

### General Form of Equation of a Circle

The general form of the equation of a circle is expressed as:

$$x^2 + y^2 + lx + my + n = 0$$

By using completing square method,

$$\left(x+\frac{l}{2}\right)^2+\left(y+\frac{m}{2}\right)^2=\frac{l^2+m^2-4n}{4}$$

Since the left side of the equation is always positive, $l^2+m^2-4n>0$.

**Note**

If $l^2+m^2-4n=0$, then the radius of the circle is zero which tells us that the circle is a point that coincides with the center. Such a type of circle is called a point circle.

If $l^2+m^2-4n<0$, then the radius of the circle become negative and not real (imaginary).

**Definition 7.3**

General form of equation of a circle is expressed as:

$\boldsymbol{x^2+y^2+lx+my+n=0}$, where $l^2+m^2-4n>0$.

**Note**

We need to make sure that the coefficients of $x^2$ and $y^2$ are 1 before applying the formula.

### Example 1

Equation of a circle is $x^2+y^2-12x-16y+19=0$. Find the center and radius of the circle.

**Solution:**

By using completing square method,

$$x^2-12x+y^2-16y+19=0$$

$$(x-6)^2+(y-8)^2-36-64+19=0$$

$$(x-6)^2+(y-8)^2=81$$

$$(x-6)^2+(y-8)^2=9^2$$

Therefore, center of the circle is $(6,8)$ and the radius of the circle is $9$ units.

### Example 2

Write the equation of the circle with center at $C(3,4)$ and that passes through the point $P(5,6)$.

**Solution:**

Let $\boldsymbol{r}$ be the radius of the circle. Then the equation of the circle is:

$(x-3)^2 + (y-4)^2 = r^2$

Since the point $P(5,6)$ is on the circle, you have

$(5-3)^2 + (6-4)^2 = r^2 \Rightarrow 2^2 + 2^2 = r^2$. Thus, $r^2 = 8$

Therefore, $(x-3)^2 + (y-4)^2 = 8$.

## Exercise 7.11

1. Find the center and radius of the circle.
   a. $x^2 + y^2 - 10x + 14y + 38 = 0$
   b. $x^2 + y^2 + 6x + 8y + 9 = 0$
2. Write the equation of the circle described below:
   a. It has center at $C(3,1)$ and pass through the point $P(7,-3)$
   b. It passes through the origin and has center at $P(-3,2)$
   c. The end point of its diameter are $A(3,4)$ and $B(-1,2)$
3. Write the general form of the circle equation with center$(-1,6)$ and radius 3 unit.
4. Write the equation of the circle in standard form $x^2 + y^2 + 8x - 2y - 8 = 0$.

## 7.6 Applications

### Area of a triangle in a coordinate plane

Area of a triangle in a coordinate geometry whose vertices are $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ is:

$$A = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

## Example 1

Find the area of the triangle having vertices at $A$, $B$, and $C$ which are at points $(-3,\ 4), (0, 1)$ and $(-3, -2)$ respectively. Also, mention the type of triangle.

**Solution:**

$$A = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$= \frac{1}{2}|(-3)(1 - (-2)) + 0(-2 - 4) + (-3)(3 - 0)|$$

$$= \frac{1}{2}|(-9) + 0 + (-9)| = 9 \text{ Sq. units}$$

$$d(AB) = \sqrt{(-3-0)^2 + (4-1)^2}$$

$$= \sqrt{18} = 3\sqrt{2}$$

$$d(BC) = \sqrt{(-3-0)^2 + (-2-1)^2}$$

$$= 3\sqrt{2}$$

So, $\overline{AB} = \overline{BC}$

And slope of $\overline{AB} = -1$ and slope of $\overline{BC} = 1$

Slope of $\overline{AB}$ $\cdot$ slope of $\overline{BC} = -1 \times 1 = -1$

Therefore, it is an isosceles right-angled triangle.

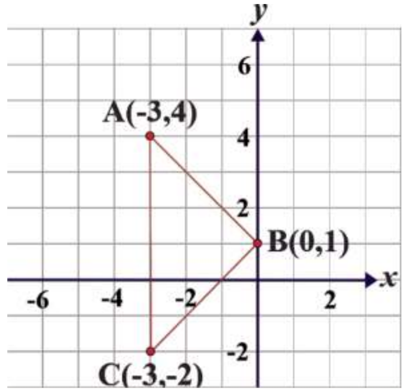



Figure 7.29

## Example 2

A triangle has vertices $A\ (-1,1), B\ (1,3)$ and $C\ (3,1)$.

**a.** Find the equation of the line containing the sides of the triangle.

**b.** Is the triangle a right-angled triangle?

**c.** What are the intercepts of the line passing through points $B$ and $C$?

**Solution:**

**a.** There are three lines of equation, AB, BC and AC.

AB: $m_1 = \frac{3-1}{1-(-1)} = \frac{2}{2} = 1$

The line passes $B(1,3)$, then $y - 3 = x - 1$ implies $y = x + 2$

BC: $m_2 = \frac{1-3}{3-1} = \frac{-2}{2} = -1.$

The line passes B(1,3), then, $y - 3 = -(x - 1)$

$y = -x + 4.$

AC: $m_3 = \frac{1-1}{3-(-1)} = \frac{0}{4} = 0$.The line passes $A(-1, 1)$, then,

$y - 1 = 0 \cdot (x + 1),\ \ y = 1$

**b.** From a. $m_1 \cdot m_2 = 1 \cdot (-1) = -1$

Thus, AB and BC are perpendicular.

Therefore, the triangle ABC is a right-angled triangle with $\angle B = 90°$.

**c.** From a., the equation of line BC: $y = -x + 4$

Thus, the $y$-intercept is 4.

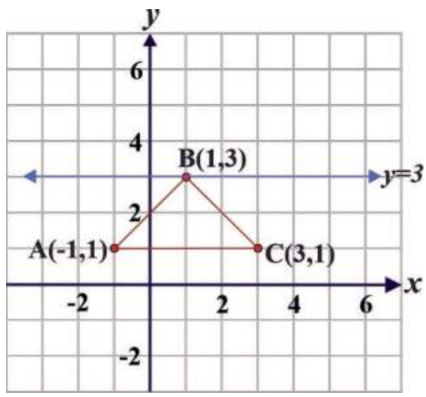



Figure 7.30

## Exercise 7.12

1. Find the area of the triangle having vertices at $A$, $B$, and $C$ which are at points $(3,\ 3), (-1,\ 0)$ and $(3, -5)$, respectively.
2. Show that the plane figure with vertices:
   a. $A(5, -1), B(2, 3)$ and $C(1, 1)$ is a right-angled triangle.
   b. $A(2, 3), B(6, 8)$ and $C(7, -1)$ is an isosceles triangle.
   c. $A(-4, 3), B(4, -3)$ and $C\ (3\sqrt{3},\ 4\sqrt{3})$ is an equilateral triangle.
3. Find the equation of the line containing side of the triangle whose vertices are $A(3, 4), B(2, 0)$ and $C(-1, 6)$.
4. Find the coordinates of a point on the $x$-axis, which is at a distance of 5 units from the point $(6, -3)$.
5. If end points of the diameter of a circle are $(-5, 2)$ and $(3, -2)$, then find the center and equation of the circle.

# Summary

1. If a point $P$ has coordinates $(a, b)$, then the number $a$ is called **the $x$-coordinate** or abscissa of $P$ and $b$ is called **the $y$-coordinate** or **ordinate of $P$**.

2. **The distance $d$ between points $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$ is given by the formula**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3. **The point $R\ (x_0, y_0)$ dividing the line segment $PQ$, internally, in the ratio $p: q$ is given by:** $R(x_0, y_0) = (\frac{px_2+qx_1}{p+q}, \frac{py_2+qy_1}{p+q})$, where $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the end-points.

   This is called the section formula.

4. **The mid-point of a line segment whose end-points are** $P(x_1, y_1)$ and $Q(x_2, y_2)$ **is given by:** $M(x_0, y_0) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$

5. **If** $P(x_1, y_1)$ and $Q(x_2, y_2)$ **are points on a line with $x_1 \neq x_2$, then the slope (Gradient) of the line, denoted by $m$, is given by** $m = \frac{y_2-y_1}{x_2-x_1}$.

6. If $\theta$ is the angle between the positive $x$-axis and the line passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ ,with $x_1 \neq x_2$ , then the slope of the line is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

7. **The graph of the equation $x = c$ is the vertical line through $P(c, 0)$ and has no slope.**

8. **The equation of the line with slope $m$ and passing through the point** $P(x_1, y_1)$ **is given by:** $y - y_1 = m(x - x_1)$

9. The equation of the line with slope $m$ and $y$-intercept $b$ is given by:

   $y = mx + b$

**10.** The equation of the line passing through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), x_1 \neq x_2$$

**11.** Two non-vertical lines are parallel if and only if they have the same slope.

**12.** **Let $l_1$ be a line with slope $m_1$ and $l_2$ be a line with slope $m_2$. Then, $l_1$ and $l_2$ are perpendicular lines if and only if $m_1 \times m_2 = -1$.**

**13.** The equation of a circle with $(h, k)$ center and $r$ radius is given by:

$$(x - h)^2 + (y - k)^2 = r^2$$

**When the center of a circle is origin, the equation is given by:** $x^2 + y^2 = r^2$

## Review Exercise

**1.** In the figure 7.31 $ABCD$ is a rectangle and its sides $AB$ and $CD$ are parallel to the $x$-axis. The coordinate of $A$ and $C$ are given in figure 7.31. Find the coordinates of $B$ and $D$.

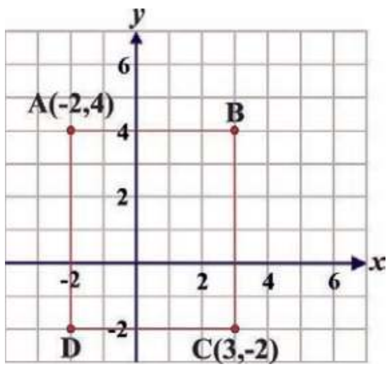



Figure 7.31

**2.** Let the coordinates of the vertices of a parallelogram be $O(0,0), A(a, 0)$ *and* $C(c, d)$.

**a.** What are the coordinates of $B$?

**b.** What are the coordinates of midpoints of $OB$ and $AC$?

**c.** What can you say from $b$?

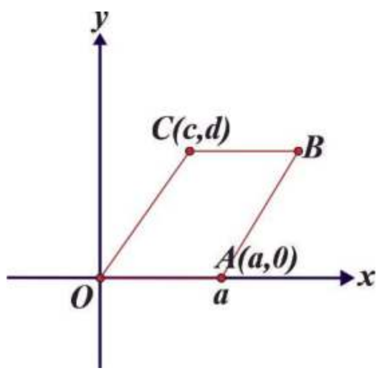



Figure 7.32

**3.** In any triangle $ABC$ prove that

$(AB)^2 + (AC)^2 = 2((AD)^2 + (DC)^2)$,

where $D$ is the midpoint of $BC$. (HINT: Let the coordinates of $B$ and $C$ be $(-a, 0)$ and $(a, 0)$, respectively. Let the coordinates of $A$ be $(d, c)$).

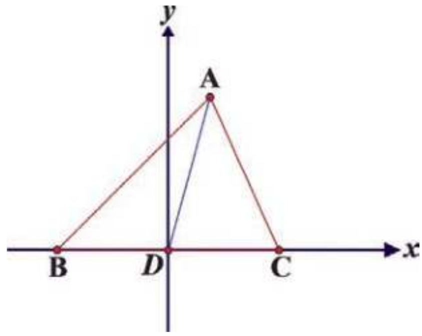



Figure 7.32

**4.** A point $C(-2, 3)$ is equidistant from points $A\,(3,\ -1)$ and $B\,(x,\ 8)$. Find the value for $x$ and the distance $BC$.

**5.** Find the ratio in which the line-segment joining the points $(5, -4)$ and $(2, 3)$ is divided by the $x$-axis.

**6.** Prove that set of points equidistant from two given points is a straight line.

**7.** If the gradient of a line is $-3$ and the $y$-intercept is $-7$, then find the equation of the line.

**8.** Find the slope of the line passing through the points $P(5, -3)$ and $Q(7, -4)$.

**9.** Find the equation of a line passing through $(-2, 3)$ and having a slope of $-1$.

**10.** Determine the equation of the line that passes through the points $A(-3, 2)$ and $B(5, 4)$.

**11.** Find the equation of the line which passes through the point $(-2, 5)$ and is perpendicular to the line whose equation is $2x - y + 5 = 0$.

**12.** If a triangle has vertices $A(-1, 1)$, $B(1, 3)$ and $C(3, 1)$, then

**a.** finds the equation of the line containing the sides of the triangle.

**b.** determines whether the triangle is a right-angled triangle or not.

**c.** What are the intercepts of the line passing through points $B$ and $C$?

**13.** Using the $x$-intercept and $y$-intercept methods, sketch the graph of

**a.** $y = 4x + 1$ **b.** $2x + 3y + 6 = 0$

**14.** Find the area of the triangle having vertices at $A$, $B$, and $C$ with coordinates $(2, 3)$, $(-1, 0)$ and $(2, -3)$, respectively. What type of triangle is it?

**15.** If one end of the diameter of a circle is $(5, 6)$ and the center of the circle is (-2, 1), then find the other end of the diameter of the circle and equation of the circle.

| | Logarithm table | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1.0 | 0 | 0.0043 | 0.0086 | 0.0128 | 0.017 | 0.0212 | 0.0253 | 0.0294 | 0.0334 | 0.0374 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 |
| 1.1 | 0.0414 | 0.0453 | 0.0492 | 0.0531 | 0.0569 | 0.0607 | 0.0645 | 0.0682 | 0.0719 | 0.0755 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 |
| 1.2 | 0.0792 | 0.0828 | 0.0864 | 0.0899 | 0.0934 | 0.0969 | 0.1004 | 0.1038 | 0.1072 | 0.1106 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 31 |
| 1.3 | 0.1139 | 0.1173 | 0.1206 | 0.1239 | 0.1271 | 0.1303 | 0.1335 | 0.1367 | 0.1399 | 0.143 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 1.4 | 0.1461 | 0.1492 | 0.1523 | 0.1553 | 0.1584 | 0.1614 | 0.1644 | 0.1673 | 0.1703 | 0.1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 1.5 | 0.1761 | 0.179 | 0.1818 | 0.1847 | 0.1875 | 0.1903 | 0.1931 | 0.1959 | 0.1987 | 0.2014 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| 1.6 | 0.2041 | 0.2068 | 0.2095 | 0.2122 | 0.2148 | 0.2175 | 0.2201 | 0.2227 | 0.2253 | 0.2279 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 |
| 1.7 | 0.2304 | 0.233 | 0.2355 | 0.238 | 0.2405 | 0.243 | 0.2455 | 0.248 | 0.2504 | 0.2529 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| 1.8 | 0.2553 | 0.2577 | 0.2601 | 0.2625 | 0.2648 | 0.2672 | 0.2695 | 0.2718 | 0.2742 | 0.2765 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 |
| 1.9 | 0.2788 | 0.281 | 0.2833 | 0.2856 | 0.2878 | 0.29 | 0.2923 | 0.2945 | 0.2967 | 0.2989 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| 2.0 | 0.301 | 0.3032 | 0.3054 | 0.3075 | 0.3096 | 0.3118 | 0.3139 | 0.316 | 0.3181 | 0.3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 2.1 | 0.3222 | 0.3243 | 0.3263 | 0.3284 | 0.3304 | 0.3324 | 0.3345 | 0.3365 | 0.3385 | 0.3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 2.2 | 0.3424 | 0.3444 | 0.3464 | 0.3483 | 0.3502 | 0.3522 | 0.3541 | 0.356 | 0.3579 | 0.3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 2.3 | 0.3617 | 0.3636 | 0.3655 | 0.3674 | 0.3692 | 0.3711 | 0.3729 | 0.3747 | 0.3766 | 0.3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 2.4 | 0.3802 | 0.382 | 0.3838 | 0.3856 | 0.3874 | 0.3892 | 0.3909 | 0.3927 | 0.3945 | 0.3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 2.5 | 0.3979 | 0.3997 | 0.4014 | 0.4031 | 0.4048 | 0.4065 | 0.4082 | 0.4099 | 0.4116 | 0.4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 2.6 | 0.415 | 0.4166 | 0.4183 | 0.42 | 0.4216 | 0.4232 | 0.4249 | 0.4265 | 0.4281 | 0.4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 2.7 | 0.4314 | 0.433 | 0.4346 | 0.4362 | 0.4378 | 0.4393 | 0.4409 | 0.4425 | 0.444 | 0.4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 2.8 | 0.4472 | 0.4487 | 0.4502 | 0.4518 | 0.4533 | 0.4548 | 0.4564 | 0.4579 | 0.4594 | 0.4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 2.9 | 0.4624 | 0.4639 | 0.4654 | 0.4669 | 0.4683 | 0.4698 | 0.4713 | 0.4728 | 0.4742 | 0.4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 3.0 | 0.4771 | 0.4786 | 0.48 | 0.4814 | 0.4829 | 0.4843 | 0.4857 | 0.4871 | 0.4886 | 0.49 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 3.1 | 0.4914 | 0.4928 | 0.4942 | 0.4955 | 0.4969 | 0.4983 | 0.4997 | 0.5011 | 0.5024 | 0.5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 3.2 | 0.5051 | 0.5065 | 0.5079 | 0.5092 | 0.5105 | 0.5119 | 0.5132 | 0.5145 | 0.5159 | 0.5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 3.3 | 0.5185 | 0.5198 | 0.5211 | 0.5224 | 0.5237 | 0.525 | 0.5263 | 0.5276 | 0.5289 | 0.5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 3.4 | 0.5315 | 0.5328 | 0.534 | 0.5353 | 0.5366 | 0.5378 | 0.5391 | 0.5403 | 0.5416 | 0.5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 3.5 | 0.5441 | 0.5453 | 0.5465 | 0.5478 | 0.549 | 0.5502 | 0.5514 | 0.5527 | 0.5539 | 0.5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 3.6 | 0.5563 | 0.5575 | 0.5587 | 0.5599 | 0.5611 | 0.5623 | 0.5635 | 0.5647 | 0.5658 | 0.567 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 3.7 | 0.5682 | 0.5694 | 0.5705 | 0.5717 | 0.5729 | 0.574 | 0.5752 | 0.5763 | 0.5775 | 0.5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3.8 | 0.5798 | 0.5809 | 0.5821 | 0.5832 | 0.5843 | 0.5855 | 0.5866 | 0.5877 | 0.5888 | 0.5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3.9 | 0.5911 | 0.5922 | 0.5933 | 0.5944 | 0.5955 | 0.5966 | 0.5977 | 0.5988 | 0.5999 | 0.601 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 4.0 | 0.6021 | 0.6031 | 0.6042 | 0.6053 | 0.6064 | 0.6075 | 0.6085 | 0.6096 | 0.6107 | 0.6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 4.1 | 0.6128 | 0.6138 | 0.6149 | 0.616 | 0.617 | 0.618 | 0.6191 | 0.6201 | 0.6212 | 0.6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4.2 | 0.6232 | 0.6243 | 0.6253 | 0.6263 | 0.6274 | 0.6284 | 0.6294 | 0.6304 | 0.6314 | 0.6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4.3 | 0.6335 | 0.6345 | 0.6355 | 0.6365 | 0.6375 | 0.6385 | 0.6395 | 0.6405 | 0.6415 | 0.6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4.4 | 0.6435 | 0.6444 | 0.6454 | 0.6464 | 0.6474 | 0.6484 | 0.6493 | 0.6503 | 0.6513 | 0.6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4.5 | 0.6532 | 0.6542 | 0.6551 | 0.6561 | 0.6571 | 0.658 | 0.659 | 0.6599 | 0.6609 | 0.6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4.6 | 0.6628 | 0.6637 | 0.6646 | 0.6656 | 0.6665 | 0.6675 | 0.6684 | 0.6693 | 0.6702 | 0.6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 4.7 | 0.6721 | 0.673 | 0.6739 | 0.6749 | 0.6758 | 0.6767 | 0.6776 | 0.6785 | 0.6794 | 0.6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 4.8 | 0.6812 | 0.6821 | 0.683 | 0.6839 | 0.6848 | 0.6857 | 0.6866 | 0.6875 | 0.6884 | 0.6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 4.9 | 0.6902 | 0.6911 | 0.692 | 0.6928 | 0.6937 | 0.6946 | 0.6955 | 0.6964 | 0.6972 | 0.6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 5.0 | 0.699 | 0.6998 | 0.7007 | 0.7016 | 0.7024 | 0.7033 | 0.7042 | 0.705 | 0.7059 | 0.7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 5.1 | 0.7076 | 0.7084 | 0.7093 | 0.7101 | 0.711 | 0.7118 | 0.7126 | 0.7135 | 0.7143 | 0.7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 5.2 | 0.716 | 0.7168 | 0.7177 | 0.7185 | 0.7193 | 0.7202 | 0.721 | 0.7218 | 0.7226 | 0.7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 5.3 | 0.7243 | 0.7251 | 0.7259 | 0.7267 | 0.7275 | 0.7284 | 0.7292 | 0.73 | 0.7308 | 0.7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 5.4 | 0.7324 | 0.7332 | 0.734 | 0.7348 | 0.7356 | 0.7364 | 0.7372 | 0.738 | 0.7388 | 0.7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 5.5 | 0.7404 | 0.7412 | 0.7419 | 0.7427 | 0.7435 | 0.7443 | 0.7451 | 0.7459 | 0.7466 | 0.7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |

| | Logarithm table | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5.6 | 0.7482 | 0.749 | 0.7497 | 0.7505 | 0.7513 | 0.752 | 0.7528 | 0.7536 | 0.7543 | 0.7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 5.7 | 0.7559 | 0.7566 | 0.7574 | 0.7582 | 0.7589 | 0.7597 | 0.7604 | 0.7612 | 0.7619 | 0.7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 5.8 | 0.7634 | 0.7642 | 0.7649 | 0.7657 | 0.7664 | 0.7672 | 0.7679 | 0.7686 | 0.7694 | 0.7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 5.9 | 0.7709 | 0.7716 | 0.7723 | 0.7731 | 0.7738 | 0.7745 | 0.7752 | 0.776 | 0.7767 | 0.7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 6.0 | 0.7782 | 0.7789 | 0.7796 | 0.7803 | 0.781 | 0.7818 | 0.7825 | 0.7832 | 0.7839 | 0.7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 6.1 | 0.7853 | 0.786 | 0.7868 | 0.7875 | 0.7882 | 0.7889 | 0.7896 | 0.7903 | 0.791 | 0.7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 6.2 | 0.7924 | 0.7931 | 0.7938 | 0.7945 | 0.7952 | 0.7959 | 0.7966 | 0.7973 | 0.798 | 0.7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 6.3 | 0.7993 | 0.8 | 0.8007 | 0.8014 | 0.8021 | 0.8028 | 0.8035 | 0.8041 | 0.8048 | 0.8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.4 | 0.8062 | 0.8069 | 0.8075 | 0.8082 | 0.8089 | 0.8096 | 0.8102 | 0.8109 | 0.8116 | 0.8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.5 | 0.8129 | 0.8136 | 0.8142 | 0.8149 | 0.8156 | 0.8162 | 0.8169 | 0.8176 | 0.8182 | 0.8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.6 | 0.8195 | 0.8202 | 0.8209 | 0.8215 | 0.8222 | 0.8228 | 0.8235 | 0.8241 | 0.8248 | 0.8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.7 | 0.8261 | 0.8267 | 0.8274 | 0.828 | 0.8287 | 0.8293 | 0.8299 | 0.8306 | 0.8312 | 0.8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.8 | 0.8325 | 0.8331 | 0.8338 | 0.8344 | 0.8351 | 0.8357 | 0.8363 | 0.837 | 0.8376 | 0.8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 6.9 | 0.8388 | 0.8395 | 0.8401 | 0.8407 | 0.8414 | 0.842 | 0.8426 | 0.8432 | 0.8439 | 0.8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 7.0 | 0.8451 | 0.8457 | 0.8463 | 0.847 | 0.8476 | 0.8482 | 0.8488 | 0.8494 | 0.85 | 0.8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 7.1 | 0.8513 | 0.8519 | 0.8525 | 0.8531 | 0.8537 | 0.8543 | 0.8549 | 0.8555 | 0.8561 | 0.8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.2 | 0.8573 | 0.8579 | 0.8585 | 0.8591 | 0.8597 | 0.8603 | 0.8609 | 0.8615 | 0.8621 | 0.8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.3 | 0.8633 | 0.8639 | 0.8645 | 0.8651 | 0.8657 | 0.8663 | 0.8669 | 0.8675 | 0.8681 | 0.8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.4 | 0.8692 | 0.8698 | 0.8704 | 0.871 | 0.8716 | 0.8722 | 0.8727 | 0.8733 | 0.8739 | 0.8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.5 | 0.8751 | 0.8756 | 0.8762 | 0.8768 | 0.8774 | 0.8779 | 0.8785 | 0.8791 | 0.8797 | 0.8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 7.6 | 0.8808 | 0.8814 | 0.882 | 0.8825 | 0.8831 | 0.8837 | 0.8842 | 0.8848 | 0.8854 | 0.8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 7.7 | 0.8865 | 0.8871 | 0.8876 | 0.8882 | 0.8887 | 0.8893 | 0.8899 | 0.8904 | 0.891 | 0.8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 7.8 | 0.8921 | 0.8927 | 0.8932 | 0.8938 | 0.8943 | 0.8949 | 0.8954 | 0.896 | 0.8965 | 0.8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 7.9 | 0.8976 | 0.8982 | 0.8987 | 0.8993 | 0.8998 | 0.9004 | 0.9009 | 0.9015 | 0.902 | 0.9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.0 | 0.9031 | 0.9036 | 0.9042 | 0.9047 | 0.9053 | 0.9058 | 0.9063 | 0.9069 | 0.9074 | 0.9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.1 | 0.9085 | 0.909 | 0.9096 | 0.9101 | 0.9106 | 0.9112 | 0.9117 | 0.9122 | 0.9128 | 0.9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.2 | 0.9138 | 0.9143 | 0.9149 | 0.9154 | 0.9159 | 0.9165 | 0.917 | 0.9175 | 0.918 | 0.9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.3 | 0.9191 | 0.9196 | 0.9201 | 0.9206 | 0.9212 | 0.9217 | 0.9222 | 0.9227 | 0.9232 | 0.9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.4 | 0.9243 | 0.9248 | 0.9253 | 0.9258 | 0.9263 | 0.9269 | 0.9274 | 0.9279 | 0.9284 | 0.9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.5 | 0.9294 | 0.9299 | 0.9304 | 0.9309 | 0.9315 | 0.932 | 0.9325 | 0.933 | 0.9335 | 0.934 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.6 | 0.9345 | 0.935 | 0.9355 | 0.936 | 0.9365 | 0.937 | 0.9375 | 0.938 | 0.9385 | 0.939 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.7 | 0.9395 | 0.94 | 0.9405 | 0.941 | 0.9415 | 0.942 | 0.9425 | 0.943 | 0.9435 | 0.944 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 8.8 | 0.9445 | 0.945 | 0.9455 | 0.946 | 0.9465 | 0.9469 | 0.9474 | 0.9479 | 0.9484 | 0.9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 8.9 | 0.9494 | 0.9499 | 0.9504 | 0.9509 | 0.9513 | 0.9518 | 0.9523 | 0.9528 | 0.9533 | 0.9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.0 | 0.9542 | 0.9547 | 0.9552 | 0.9557 | 0.9562 | 0.9566 | 0.9571 | 0.9576 | 0.9581 | 0.9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.1 | 0.959 | 0.9595 | 0.96 | 0.9605 | 0.9609 | 0.9614 | 0.9619 | 0.9624 | 0.9628 | 0.9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.2 | 0.9638 | 0.9643 | 0.9647 | 0.9652 | 0.9657 | 0.9661 | 0.9666 | 0.9671 | 0.9675 | 0.968 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.3 | 0.9685 | 0.9689 | 0.9694 | 0.9699 | 0.9703 | 0.9708 | 0.9713 | 0.9717 | 0.9722 | 0.9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.4 | 0.9731 | 0.9736 | 0.9741 | 0.9745 | 0.975 | 0.9754 | 0.9759 | 0.9763 | 0.9768 | 0.9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.5 | 0.9777 | 0.9782 | 0.9786 | 0.9791 | 0.9795 | 0.98 | 0.9805 | 0.9809 | 0.9814 | 0.9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.6 | 0.9823 | 0.9827 | 0.9832 | 0.9836 | 0.9841 | 0.9845 | 0.985 | 0.9854 | 0.9859 | 0.9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.7 | 0.9868 | 0.9872 | 0.9877 | 0.9881 | 0.9886 | 0.989 | 0.9894 | 0.9899 | 0.9903 | 0.9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.8 | 0.9912 | 0.9917 | 0.9921 | 0.9926 | 0.993 | 0.9934 | 0.9939 | 0.9943 | 0.9948 | 0.9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.9 | 0.9956 | 0.9961 | 0.9965 | 0.9969 | 0.9974 | 0.9978 | 0.9983 | 0.9987 | 0.9991 | 0.9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |

| | Anti Logarithm table | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.00 | 1 | 1.002 | 1.005 | 1.007 | 1.009 | 1.012 | 1.014 | 1.016 | 1.019 | 1.021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0.01 | 1.023 | 1.026 | 1.028 | 1.03 | 1.033 | 1.035 | 1.038 | 1.04 | 1.042 | 1.045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0.02 | 1.047 | 1.05 | 1.052 | 1.054 | 1.057 | 1.059 | 1.062 | 1.064 | 1.067 | 1.069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0.03 | 1.072 | 1.074 | 1.076 | 1.079 | 1.081 | 1.084 | 1.086 | 1.089 | 1.091 | 1.094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0.04 | 1.096 | 1.099 | 1.102 | 1.104 | 1.107 | 1.109 | 1.112 | 1.114 | 1.117 | 1.119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 0.05 | 1.122 | 1.125 | 1.127 | 1.13 | 1.132 | 1.135 | 1.138 | 1.14 | 1.143 | 1.146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 0.06 | 1.148 | 1.151 | 1.153 | 1.156 | 1.159 | 1.161 | 1.164 | 1.167 | 1.169 | 1.172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 0.07 | 1.175 | 1.178 | 1.18 | 1.183 | 1.186 | 1.189 | 1.191 | 1.194 | 1.197 | 1.199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 0.08 | 1.202 | 1.205 | 1.208 | 1.211 | 1.213 | 1.216 | 1.219 | 1.222 | 1.225 | 1.227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 0.09 | 1.23 | 1.233 | 1.236 | 1.239 | 1.242 | 1.245 | 1.247 | 1.25 | 1.253 | 1.256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 0.10 | 1.259 | 1.262 | 1.265 | 1.268 | 1.271 | 1.274 | 1.276 | 1.279 | 1.282 | 1.285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 0.11 | 1.288 | 1.291 | 1.294 | 1.297 | 1.3 | 1.303 | 1.306 | 1.309 | 1.312 | 1.315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| 0.12 | 1.318 | 1.321 | 1.324 | 1.327 | 1.33 | 1.334 | 1.337 | 1.34 | 1.343 | 1.346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| 0.13 | 1.349 | 1.352 | 1.355 | 1.358 | 1.361 | 1.365 | 1.368 | 1.371 | 1.374 | 1.377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 0.14 | 1.38 | 1.384 | 1.387 | 1.39 | 1.393 | 1.396 | 1.4 | 1.403 | 1.406 | 1.409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 0.15 | 1.413 | 1.416 | 1.419 | 1.422 | 1.426 | 1.429 | 1.432 | 1.435 | 1.439 | 1.442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 0.16 | 1.445 | 1.449 | 1.452 | 1.455 | 1.459 | 1.462 | 1.466 | 1.469 | 1.472 | 1.476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 0.17 | 1.479 | 1.483 | 1.486 | 1.489 | 1.493 | 1.496 | 1.5 | 1.503 | 1.507 | 1.51 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 0.18 | 1.514 | 1.517 | 1.521 | 1.524 | 1.528 | 1.531 | 1.535 | 1.538 | 1.542 | 1.545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 0.19 | 1.549 | 1.552 | 1.556 | 1.56 | 1.563 | 1.567 | 1.57 | 1.574 | 1.578 | 1.581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| 0.20 | 1.585 | 1.589 | 1.592 | 1.596 | 1.6 | 1.603 | 1.607 | 1.611 | 1.614 | 1.618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| 0.21 | 1.622 | 1.626 | 1.629 | 1.633 | 1.637 | 1.641 | 1.644 | 1.648 | 1.652 | 1.656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| 0.22 | 1.66 | 1.663 | 1.667 | 1.671 | 1.675 | 1.679 | 1.683 | 1.687 | 1.69 | 1.694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| 0.23 | 1.698 | 1.702 | 1.706 | 1.71 | 1.714 | 1.718 | 1.722 | 1.726 | 1.73 | 1.734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 0.24 | 1.738 | 1.742 | 1.746 | 1.75 | 1.754 | 1.758 | 1.762 | 1.766 | 1.77 | 1.774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 0.25 | 1.778 | 1.782 | 1.786 | 1.791 | 1.795 | 1.799 | 1.803 | 1.807 | 1.811 | 1.816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 0.26 | 1.82 | 1.824 | 1.828 | 1.832 | 1.837 | 1.841 | 1.845 | 1.849 | 1.854 | 1.858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 0.27 | 1.862 | 1.866 | 1.871 | 1.875 | 1.879 | 1.884 | 1.888 | 1.892 | 1.897 | 1.901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 0.28 | 1.905 | 1.91 | 1.914 | 1.919 | 1.923 | 1.928 | 1.932 | 1.936 | 1.941 | 1.945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 0.29 | 1.95 | 1.954 | 1.959 | 1.963 | 1.968 | 1.972 | 1.977 | 1.982 | 1.986 | 1.991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 0.30 | 1.995 | 2 | 2.004 | 2.009 | 2.014 | 2.018 | 2.023 | 2.028 | 2.032 | 2.037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 0.31 | 2.042 | 2.046 | 2.051 | 2.056 | 2.061 | 2.065 | 2.07 | 2.075 | 2.08 | 2.084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 0.32 | 2.089 | 2.094 | 2.099 | 2.104 | 2.109 | 2.113 | 2.118 | 2.123 | 2.128 | 2.133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 0.33 | 2.138 | 2.143 | 2.148 | 2.153 | 2.158 | 2.163 | 2.168 | 2.173 | 2.178 | 2.183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 0.34 | 2.188 | 2.193 | 2.198 | 2.203 | 2.208 | 2.213 | 2.218 | 2.223 | 2.228 | 2.234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 0.35 | 2.239 | 2.244 | 2.249 | 2.254 | 2.259 | 2.265 | 2.27 | 2.275 | 2.28 | 2.286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 0.36 | 2.291 | 2.296 | 2.301 | 2.307 | 2.312 | 2.317 | 2.323 | 2.328 | 2.333 | 2.339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 0.37 | 2.344 | 2.35 | 2.355 | 2.36 | 2.366 | 2.371 | 2.377 | 2.382 | 2.388 | 2.393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 0.38 | 2.399 | 2.404 | 2.41 | 2.415 | 2.421 | 2.427 | 2.432 | 2.438 | 2.443 | 2.449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 0.39 | 2.455 | 2.46 | 2.466 | 2.472 | 2.477 | 2.483 | 2.489 | 2.495 | 2.5 | 2.506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 0.40 | 2.512 | 2.518 | 2.523 | 2.529 | 2.535 | 2.541 | 2.547 | 2.553 | 2.559 | 2.564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 0.41 | 2.57 | 2.576 | 2.582 | 2.588 | 2.594 | 2.6 | 2.606 | 2.612 | 2.618 | 2.624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 0.42 | 2.63 | 2.636 | 2.642 | 2.649 | 2.655 | 2.661 | 2.667 | 2.673 | 2.679 | 2.685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 0.43 | 2.692 | 2.698 | 2.704 | 2.71 | 2.716 | 2.723 | 2.729 | 2.735 | 2.742 | 2.748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 0.44 | 2.754 | 2.761 | 2.767 | 2.773 | 2.78 | 2.786 | 2.793 | 2.799 | 2.805 | 2.812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 0.45 | 2.818 | 2.825 | 2.831 | 2.838 | 2.844 | 2.851 | 2.858 | 2.864 | 2.871 | 2.877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 0.46 | 2.884 | 2.891 | 2.897 | 2.904 | 2.911 | 2.917 | 2.924 | 2.931 | 2.938 | 2.944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 0.47 | 2.951 | 2.958 | 2.965 | 2.972 | 2.979 | 2.985 | 2.992 | 2.999 | 3.006 | 3.013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 0.48 | 3.02 | 3.027 | 3.034 | 3.041 | 3.048 | 3.055 | 3.062 | 3.069 | 3.076 | 3.083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 0.49 | 3.09 | 3.097 | 3.105 | 3.112 | 3.119 | 3.126 | 3.133 | 3.141 | 3.148 | 3.155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 0.50 | 3.162 | 3.17 | 3.177 | 3.184 | 3.192 | 3.199 | 3.206 | 3.214 | 3.221 | 3.228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |

| | Anti Logarithm table | | | | | | | | | | Mean Difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.51 | 3.236 | 3.243 | 3.251 | 3.258 | 3.266 | 3.273 | 3.281 | 3.289 | 3.296 | 3.304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 0.52 | 3.311 | 3.319 | 3.327 | 3.334 | 3.342 | 3.35 | 3.357 | 3.365 | 3.373 | 3.381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 0.53 | 3.388 | 3.396 | 3.404 | 3.412 | 3.42 | 3.428 | 3.436 | 3.443 | 3.451 | 3.459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 0.54 | 3.467 | 3.475 | 3.483 | 3.491 | 3.499 | 3.508 | 3.516 | 3.524 | 3.532 | 3.54 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 0.55 | 3.548 | 3.556 | 3.565 | 3.573 | 3.581 | 3.589 | 3.597 | 3.606 | 3.614 | 3.622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 0.56 | 3.631 | 3.639 | 3.648 | 3.656 | 3.664 | 3.673 | 3.681 | 3.69 | 3.698 | 3.707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.57 | 3.715 | 3.724 | 3.733 | 3.741 | 3.75 | 3.758 | 3.767 | 3.776 | 3.784 | 3.793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.58 | 3.802 | 3.811 | 3.819 | 3.828 | 3.837 | 3.846 | 3.855 | 3.864 | 3.873 | 3.882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 0.59 | 3.89 | 3.899 | 3.908 | 3.917 | 3.926 | 3.936 | 3.945 | 3.954 | 3.963 | 3.972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 0.60 | 3.981 | 3.99 | 3.999 | 4.009 | 4.018 | 4.027 | 4.036 | 4.046 | 4.055 | 4.064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| 0.61 | 4.074 | 4.083 | 4.093 | 4.102 | 4.111 | 4.121 | 4.13 | 4.14 | 4.15 | 4.159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.62 | 4.169 | 4.178 | 4.188 | 4.198 | 4.207 | 4.217 | 4.227 | 4.236 | 4.246 | 4.256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.63 | 4.266 | 4.276 | 4.285 | 4.295 | 4.305 | 4.315 | 4.325 | 4.335 | 4.345 | 4.355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.64 | 4.365 | 4.375 | 4.385 | 4.395 | 4.406 | 4.416 | 4.426 | 4.436 | 4.446 | 4.457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.65 | 4.467 | 4.477 | 4.487 | 4.498 | 4.508 | 4.519 | 4.529 | 4.539 | 4.55 | 4.56 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0.66 | 4.571 | 4.581 | 4.592 | 4.603 | 4.613 | 4.624 | 4.634 | 4.645 | 4.656 | 4.667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| 0.67 | 4.677 | 4.688 | 4.699 | 4.71 | 4.721 | 4.732 | 4.742 | 4.753 | 4.764 | 4.775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 0.68 | 4.786 | 4.797 | 4.808 | 4.819 | 4.831 | 4.842 | 4.853 | 4.864 | 4.875 | 4.887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| 0.69 | 4.898 | 4.909 | 4.92 | 4.932 | 4.943 | 4.955 | 4.966 | 4.977 | 4.989 | 5 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0.70 | 5.012 | 5.023 | 5.035 | 5.047 | 5.058 | 5.07 | 5.082 | 5.093 | 5.105 | 5.117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| 0.71 | 5.129 | 5.14 | 5.152 | 5.164 | 5.176 | 5.188 | 5.2 | 5.212 | 5.224 | 5.236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 0.72 | 5.248 | 5.26 | 5.272 | 5.284 | 5.297 | 5.309 | 5.321 | 5.333 | 5.346 | 5.358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 0.73 | 5.37 | 5.383 | 5.395 | 5.408 | 5.42 | 5.433 | 5.445 | 5.458 | 5.47 | 5.483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 0.74 | 5.495 | 5.508 | 5.521 | 5.534 | 5.546 | 5.559 | 5.572 | 5.585 | 5.598 | 5.61 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 0.75 | 5.623 | 5.636 | 5.649 | 5.662 | 5.675 | 5.689 | 5.702 | 5.715 | 5.728 | 5.741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| 0.76 | 5.754 | 5.768 | 5.781 | 5.794 | 5.808 | 5.821 | 5.834 | 5.848 | 5.861 | 5.875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 0.77 | 5.888 | 5.902 | 5.916 | 5.929 | 5.943 | 5.957 | 5.97 | 5.984 | 5.998 | 6.012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| 0.78 | 6.026 | 6.039 | 6.053 | 6.067 | 6.081 | 6.095 | 6.109 | 6.124 | 6.138 | 6.152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 0.79 | 6.166 | 6.18 | 6.194 | 6.209 | 6.223 | 6.237 | 6.252 | 6.266 | 6.281 | 6.295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 0.80 | 6.31 | 6.324 | 6.339 | 6.353 | 6.368 | 6.383 | 6.397 | 6.412 | 6.427 | 6.442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 0.81 | 6.457 | 6.471 | 6.486 | 6.501 | 6.516 | 6.531 | 6.546 | 6.561 | 6.577 | 6.592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 0.82 | 6.607 | 6.622 | 6.637 | 6.653 | 6.668 | 6.683 | 6.699 | 6.714 | 6.73 | 6.745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 0.83 | 6.761 | 6.776 | 6.792 | 6.808 | 6.823 | 6.839 | 6.855 | 6.871 | 6.887 | 6.902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 0.84 | 6.918 | 6.934 | 6.95 | 6.966 | 6.982 | 6.998 | 7.015 | 7.031 | 7.047 | 7.063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| 0.85 | 7.079 | 7.096 | 7.112 | 7.129 | 7.145 | 7.161 | 7.178 | 7.194 | 7.211 | 7.228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| 0.86 | 7.244 | 7.261 | 7.278 | 7.295 | 7.311 | 7.328 | 7.345 | 7.362 | 7.379 | 7.396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| 0.87 | 7.413 | 7.43 | 7.447 | 7.464 | 7.482 | 7.499 | 7.516 | 7.534 | 7.551 | 7.568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| 0.88 | 7.586 | 7.603 | 7.621 | 7.638 | 7.656 | 7.674 | 7.691 | 7.709 | 7.727 | 7.745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 0.89 | 7.762 | 7.78 | 7.798 | 7.816 | 7.834 | 7.852 | 7.87 | 7.889 | 7.907 | 7.925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| 0.90 | 7.943 | 7.962 | 7.98 | 7.998 | 8.017 | 8.035 | 8.054 | 8.072 | 8.091 | 8.11 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 0.91 | 8.128 | 8.147 | 8.166 | 8.185 | 8.204 | 8.222 | 8.241 | 8.26 | 8.279 | 8.299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| 0.92 | 8.318 | 8.337 | 8.356 | 8.375 | 8.395 | 8.414 | 8.433 | 8.453 | 8.472 | 8.492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 0.93 | 8.511 | 8.531 | 8.551 | 8.57 | 8.59 | 8.61 | 8.63 | 8.65 | 8.67 | 8.69 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 0.94 | 8.71 | 8.73 | 8.75 | 8.77 | 8.79 | 8.81 | 8.831 | 8.851 | 8.872 | 8.892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 0.95 | 8.913 | 8.933 | 8.954 | 8.974 | 8.995 | 9.016 | 9.036 | 9.057 | 9.078 | 9.099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| 0.96 | 9.12 | 9.141 | 9.162 | 9.183 | 9.204 | 9.226 | 9.247 | 9.268 | 9.29 | 9.311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 0.97 | 9.333 | 9.354 | 9.376 | 9.397 | 9.419 | 9.441 | 9.462 | 9.484 | 9.506 | 9.528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| 0.98 | 9.55 | 9.572 | 9.594 | 9.616 | 9.638 | 9.661 | 9.683 | 9.705 | 9.727 | 9.75 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| 0.99 | 9.772 | 9.795 | 9.817 | 9.84 | 9.863 | 9.886 | 9.908 | 9.931 | 9.954 | 9.977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

| | sin | cos | tan | cot | sec | csc | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | | 1 | | 90 |
| 1 | 0.017452 | 0.999848 | 0.017455 | 57.2900 | 1.000152 | 57.29874 | 89 |
| 2 | 0.03490 | 0.999391 | 0.034921 | 28.63628 | 1.00061 | 28.65373 | 88 |
| 3 | 0.052336 | 0.99863 | 0.052408 | 19.08115 | 1.001372 | 19.10734 | 87 |
| 4 | 0.069756 | 0.997564 | 0.069927 | 14.30068 | 1.002442 | 14.3356 | 86 |
| 5 | 0.087156 | 0.996195 | 0.087489 | 11.43006 | 1.00382 | 11.47372 | 85 |
| 6 | 0.104528 | 0.994522 | 0.10510 | 9.514373 | 1.005508 | 9.56678 | 84 |
| 7 | 0.121869 | 0.992546 | 0.122784 | 8.144353 | 1.00751 | 8.205516 | 83 |
| 8 | 0.139173 | 0.990268 | 0.140541 | 7.115376 | 1.009828 | 7.18530 | 82 |
| 9 | 0.156434 | 0.987688 | 0.158384 | 6.313757 | 1.012465 | 6.392459 | 81 |
| 10 | 0.173648 | 0.984808 | 0.176327 | 5.671287 | 1.015427 | 5.758775 | 80 |
| 11 | 0.190809 | 0.981627 | 0.19438 | 5.144558 | 1.018717 | 5.240847 | 79 |
| 12 | 0.207912 | 0.978148 | 0.212556 | 4.704634 | 1.022341 | 4.809738 | 78 |
| 13 | 0.224951 | 0.97437 | 0.230868 | 4.33148 | 1.026304 | 4.445415 | 77 |
| 14 | 0.241922 | 0.97030 | 0.249328 | 4.010784 | 1.030614 | 4.133569 | 76 |
| 15 | 0.258819 | 0.965926 | 0.267949 | 3.732054 | 1.035276 | 3.863706 | 75 |
| 16 | 0.275637 | 0.961262 | 0.286745 | 3.487418 | 1.040299 | 3.627958 | 74 |
| 17 | 0.292371 | 0.95630 | 0.30573 | 3.270856 | 1.045692 | 3.420306 | 73 |
| 18 | 0.309017 | 0.951057 | 0.324919 | 3.077686 | 1.051462 | 3.236071 | 72 |
| 19 | 0.325568 | 0.945519 | 0.344327 | 2.904214 | 1.057621 | 3.071556 | 71 |
| 20 | 0.34202 | 0.939693 | 0.36397 | 2.74748 | 1.064178 | 2.923807 | 70 |
| 21 | 0.358368 | 0.933581 | 0.383864 | 2.605091 | 1.071145 | 2.79043 | 69 |
| 22 | 0.374606 | 0.927184 | 0.404026 | 2.475089 | 1.078535 | 2.669469 | 68 |
| 23 | 0.390731 | 0.920505 | 0.424474 | 2.355855 | 1.08636 | 2.559307 | 67 |
| 24 | 0.406736 | 0.913546 | 0.445228 | 2.246039 | 1.094636 | 2.45860 | 66 |
| 25 | 0.422618 | 0.906308 | 0.466307 | 2.144509 | 1.103378 | 2.36620 | 65 |
| 26 | 0.438371 | 0.898794 | 0.487732 | 2.050306 | 1.112602 | 2.281174 | 64 |
| 27 | 0.45399 | 0.891007 | 0.509525 | 1.962612 | 1.122326 | 2.202691 | 63 |
| 28 | 0.469471 | 0.882948 | 0.531709 | 1.880728 | 1.13257 | 2.130056 | 62 |
| 29 | 0.484809 | 0.87462 | 0.554308 | 1.80405 | 1.143354 | 2.062667 | 61 |
| 30 | 0.5000 | 0.866026 | 0.57735 | 1.732053 | 1.1547 | 2.00000 | 60 |
| 31 | 0.515038 | 0.857168 | 0.60086 | 1.664281 | 1.166633 | 1.941606 | 59 |
| 32 | 0.529919 | 0.848048 | 0.624869 | 1.600336 | 1.179178 | 1.887081 | 58 |
| 33 | 0.544639 | 0.838671 | 0.649407 | 1.539867 | 1.192363 | 1.83608 | 57 |
| 34 | 0.559192 | 0.829038 | 0.674508 | 1.482563 | 1.206218 | 1.788293 | 56 |
| 35 | 0.573576 | 0.819152 | 0.700207 | 1.42815 | 1.220774 | 1.743448 | 55 |
| 36 | 0.587785 | 0.809017 | 0.726542 | 1.376383 | 1.236068 | 1.70130 | 54 |
| 37 | 0.601815 | 0.798636 | 0.753553 | 1.327046 | 1.252135 | 1.661641 | 53 |
| 38 | 0.615661 | 0.788011 | 0.781285 | 1.279943 | 1.269018 | 1.62427 | 52 |
| 39 | 0.62932 | 0.777146 | 0.809783 | 1.23490 | 1.286759 | 1.589017 | 51 |
| 40 | 0.642787 | 0.766045 | 0.83910 | 1.191755 | 1.305407 | 1.555725 | 50 |
| 41 | 0.656059 | 0.75471 | 0.869286 | 1.15037 | 1.325012 | 1.524254 | 49 |
| 42 | 0.66913 | 0.743145 | 0.90040 | 1.110614 | 1.345632 | 1.494478 | 48 |
| 43 | 0.68200 | 0.731354 | 0.932514 | 1.07237 | 1.367327 | 1.46628 | 47 |
| 44 | 0.694658 | 0.71934 | 0.965688 | 1.035532 | 1.390163 | 1.439558 | 46 |
| 45 | 0.707106 | 0.707107 | 1.00000 | 1.00000 | 1.414213 | 1.414215 | 45 |
| | cos | sin | cot | tan | csc | sec | |